Preliminary Mathematics

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PRELIMINARY MATHEMATICS

BY

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PREFACE

The ultimate aims in the study of mathematics may be to obtain a practical knowledge of numbers or to discipline the mental powers. The realization of either aim demands calculation, and calculation is an art.

That book which best assists in the realization of knowledge of numbers or in mental training must be comprehensive in principles and extensive in details. The author of this book does not expect that this book will displace fiction or relieve the student of occasion for mental effort.

For many generations Algebra has followed Arithmetic in educational and scientific courses, and the change from a study of numbers to a system of reasoning solely conducted by symbols and letters, gives the average student such an educational jolt as to induce discouragement, and largely contributes to the cessation of school education at or before the high school stage.

If this small book should prove in a limited degree a connecting link between the study of Arithmetic and the study of Algebra for the boys and girls of any country, the writer would consider the labor of preparation well repaid.

While this book was designed originally for the use of those whose educational training had been limited, it has been so remodelled as to adapt it for the use of pupils attending the eighth grade and the high school, or the "Junior High School."

The subject matter up to page 77 is suitable for pupils in the eighth grade and below, while the remaining portion of the text will prove of assistance to pupils in the high schools.

As this book is to be used as an auxiliary, in conjunction with other text books, many points are explained herein that are passed over in ordinary text books. The chief object of this book is to *show how to solve problems.*

Many of the examples and problems presented in this book are original, having been evolved in the process of many years of teaching; others have been taken from standard text books, and some have been received for solution from students studying to pass college entrance examinations.

One aim in the preparation of this book was to show practical applications of the theory discussed.

The value of the book as a reference book will be found to be greatly increased by the insertion of a comprehensive *index, referring directly to pages.*

Useful tables pertaining to interest, and to weights and measures have been added to the subject matter.

HOW TO STUDY

With mental, as with physical development, success depends upon regular, systematic training. The athlete to become successful, is obliged to undergo a certain period of physical training, consisting of certain forms of muscular exercise, repeated regularly every day; of a regular diet consisting of carefully selected foods, and an abstinence from certain actions, drugs and intoxicating liquors that have been proved, by scientific investigation to be detrimental to the human system.

It seems logical that those features adopted in the training for physical development, should also be adopted in training for mental development.

When training for physical development, it is usual not to neglect the mental condition; as is evidenced in the training of college athletes and professionals.

On the other hand it is not advisable for those who are undergoing mental training, to neglect their physical welfare.

Health is wealth,—not to be evaluated in the term of dollars. Active healthy brains are not likely to continue in efficient operation in unhealthy bodies.

The fundamental principle underlying the college education is constant systematic *application*. It is this required attention to daily lessons that makes a college course a success in the development of an educated person.

Any educational system to be in any degree a successful one, must be logically arranged. Some college courses are more successful and beneficial than are others, simply because of the logical sequence of their *subjects* and studies.

SUCCESS is not a *talent* but an ACQUISITION.

All great achievements and notable professional careers, have been possible only as the result of continuous application, in the building up or assembling, day by day, detail upon detail, lesson upon lesson, experiment after experiment.

The great author simply assembles a multitude of individual *ideas* in the form of a story or a book.

The noted artist assembles his separate ideas on the canvas in the shape of a painting.

The inventor has all the characteristics of the author and painter in the assembling of his ideas of the application of the laws of Physics, Mechanics, Chemistry, and Electricity, to make a complete, useful, properly operating, and valuable device.

The greatness of all of these, results from bringing together, for harmonious comprehension, a number of single ideas, or facts, or detached portions of scientific knowledge.

In studying any scientific subject one should begin in a systematic and logical manner; should observe extreme care to *study a certain definite amount each day;* even if no longer than a half-hour; *should each day learn one new fact* pertaining to the subject, and should constantly think of various applications of each new fact; either as applied to some invention, or as applied in the economic operations of daily life.

Above all other considerations of importance, one should never allow any event, except perchance one's own death, to interrupt the continuity of this daily routine.

By proper daily routine it lies within the power of any individual to make whatever one pleases of one's self.

The *incentive* to become a useful citizen or a noted professional man must come from *within* the individual. *Nobody can kick goodness or greatness into you.* The most noted teacher may teach you each day during four years, but you may not learn anything as a result. A teacher cannot make you *learn* a subject if you decide not to. Each individual usually has *preference* as to the kind of work to be engaged in during life. After one decides what particular branch to pursue, then should be formulated the resolution to apply one's energies to become proficient in this special line.

The method adopted in the presentation of these courses is such as to arrive at results in the very shortest time possible, without sacrificing accuracy or clearness. This may perforce mean that some students, because of limited educational advantages in early life, will find it necessary to consult other books dealing with certain fundamental processes in Arithmetic, Algebra, Geometry, and Physics. Wherever such consultation is considered timely or valuable throughout the courses, the names of desirable books will be mentioned.

The purchase of these books will result in the possession of a valuable reference library.

The predominating idea in all engineering progress is *utility* and this idea is prominent throughout this course.

GENERAL INSTRUCTIONS

All *general instructions* should be read carefully several times before engaging in any work that is to be submitted, and should be carefully consulted when in doubt as to the proper method to pursue.

The calculations of all problems should be plainly copied, with ink, into the proper blank spaces reserved for them throughout the book.

The arrangements of the given "data" and the successive steps in the mathematical process may be observed in **Example 7**, page 32, and in **Example 15**, page 68. These *Examples* should serve as guides in solving other problems.

Always read the directions and rules applying to mathematical processes several times very carefully before proceeding to perform calculations.

A problem may be defined as a question proposed, to which an intelligent answer is desired. A problem is to be *solved* or to be *worked out*. The statement of the problem usually lays down certain conditions, or states certain relations that exist between certain *known* quantities. The problem is solved by obtaining the numerical value of (or the relation of) one or more *unknown* quantities, from the known quantities and their relation with the unknown.

A problem is very different from an *example*.

An example is a problem so solved as to show the process of solution, step by step, serving as a guide in the solution of problems of a similar nature.

Every individual is a problem; both a physical and a psychological one, but not every individual can be said to be either a physical or a psychological *example:* for others to copy.

DIRECTIONS APPLYING TO WORK IN PRELIMINARY MATHEMATICS

PRELIMINARY MATHEMATICS is divided into *lessons,* each of proper length to be studied and learned in about two hours by a student of ordinary ability. Concentrated thought and continuous application will accomplish more than quickness and flashy brilliancy.

Complete all of the **problems,** some marked with numerals; some with numerals and letters, **before beginning the study of a succeeding lesson.**

The lessons are of increasing difficulty as they continue; but the last lessons will require no more time than the first, if the regular sequence is adhered to.

It will be better to solve the problems on scrap paper, using a pencil, and then carefully copy the complete solution step by step with *ink,* in the spaces left blank for them throughout the book.

THE GREEK ALPHABET

Symbol	Name in English	English Equivalent
Α α	Alpha	A a
Β β	Beta	B b
Γ γ	Gamma	G g
Δ δ	Delta	D d
Ε ϵ	Epsilon	E e (short)
Ζ ζ	Zeta	Z z
Η η	Eta	E e (long)
Θ θ	Theta	Th
Ι ι	Iota	I i
Κ κ	Kappa	K k (or hard c)
Λ λ	Lambda	L l
Μ μ	Mu	M m
Ν ν	Nu	N n
Ξ ξ	Xi	X x
Ο o	Omicron	O o (short)
Π π	Pi	P p
Ρ ρ	Rho	R r
Σ σ	Sigma	S s
Τ τ	Tau	T t
Υ υ	Upsilon	(u) y
Φ ϕ	Phi	Ph
Χ χ	Chi	Kh
Ψ ψ	Psi	Ps
Ω ω	Omega	O o (long)

INTRODUCTORY

Quantity; Measurement; Number

QUANTITY is that which is capable of being increased, diminished, and measured; such as *time, distance,* and *weight*: the three most important quantities which enter into our common experience. From these fundamental quantities are obtained other important quantities, as *area, volume, energy, work* and *power.*

Any quantity may be considered as made up of a number of smaller amounts of the same quantity. These smaller amounts are called *units.*

A quantity is measured by the process of *finding how many times the quantity contains the* smaller quantity or *unit.*

In measuring any quantity, the unit used in the process, must be of the same *kind* as the quantity itself.

The expression denoting a quantity usually consists of two parts or symbols; one part or symbol denotes the *amount* of the quantity; while the other part denotes the *kind.*

The symbol denoting the amount is sometimes called the *numeric* or the coefficient.

Examples: 10 seconds; 5 miles; 2000 pounds; 10 apples; 5 amperes; 110 volts; all of these quantities named are measurable quantities and are called *concrete* quantities.

It may be said then that *the relation between a quantity and its unit is an abstract number.* In the preceding examples, the coefficients 10, 5 and 110 are abstract numbers. An abstract number denotes some operation or process of measuring, and is sometimes alluded to as an operator.

ALL UNITS ARE ARBITRARILY ASSUMED. Some units have been chosen with some degree of reason while others have been assumed without much reason.

The so-called *"English"* system of units contains units that have been wisely chosen as regards convenience, being in this respect far more desirable than the so-called "Metric" system. The only point worthy of consideration that has been urged in favor of the Metric system is that it is a *decimal* system; that is, the units increase by ten or multiples of ten, and decrease by *"tenths"* or sub-multiples of ten.

The fundamental unit in the English system is the *foot,* which is a very convenient length for all common transactions in our daily experience.

The foot being divided into *twelve* equal parts gives *the inch* which *is a very convenient length* for purposes for which the foot is too long.

The yard, which is three feet in length is also a very convenient unit.

The fundamental unit in the Metric system is the *meter,* which is equal to very nearly (ninety-one and four-tenths), 91.4 inches.

The one-hundredth part of a meter is a small unit called the centimeter, which is altogether too small for ordinary uses. The decimeter would be one-tenth of a meter, or ten centimeters, which is a very inconvenient length for ordinary use.

The argument that 10 is a more *convenient* number for common use than 12, is questionable. For example the number 10 is divisible by 1, 2, 5 and 10, an even number of times without a remainder: The number 12 is divisible by 1, 2, 3, 4, 6 and 12, without a remainder: two more divisors than 10 has. Take the number 36 (36 inches in a yard). This number is divisible by 1, 2, 3, 4, 6, 9, 12, 18, and 36, without a remainder. The number 100 is divisible by 1, 2, 4, 5, 10, 20, 25, 50 and 100. Here is a number nearly three times as large as 36, which only has the same number of divisors. The number 72, which is twice 36, has twelve divisors.

As regards performing mathematical operations, if operations should be performed, employing *common fractions,* much labor would be saved, and more satisfactory results obtained than when the *decimal* system is employed, with its repeating decimal fraction.

Examples of this will be given from time to time, which should be carefully noted.

As a matter of fact each country fixes by law the standard units for the use of its inhabitants.

The standard units are kept very carefully either at the capitol or at an important city of each country.

For example, the yard is fixed by United States law as the distance between two marks on a certain bar of metal, when at a certain temperature. One-third of this standard distance is called one foot.

The yard is the *standard* of length, but the FOOT is the common *unit* of length.

Standards of volume and weight are fixed in a similar manner.

While the use of the metric system of units is *legal* in the United States the system is not the commonly adopted system.

The standard length, adopted by the French, is called the meter; which in reality is the distance between two marks on a certain bar of platinum-iridium, carefully kept in the International Metric Bureau at Sèvres, near Paris.

The commonly adopted *unit* of length is $\frac{1}{100}$ of the meter, and is called a *centimeter.* $\frac{1}{1000}$ of a meter is called a millimeter. $\frac{1}{1000}$ of a meter is $\frac{1}{10}$ of a centimeter.

LESSON I

SYMBOLS OF NUMBER AND NOTATION

The symbols usually employed to denote *number* are the *Arabic numerals,* 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, and the letters of the English alphabet. The Arabic numerals or characters are alluded to as *figures;* sometimes called *digits.* The tenth character as printed above, is called "*cipher*" or "*zero*", and denotes a quantity too small to be measured. In direct contrast to 0 another symbol, ∞ is used to denote a quantity that is *too large* to be measured; called *infinity.* The sign or symbol ∞ denotes a quantity greater than any that can be denoted by any arrangement of the Arabic numerals.

The numerals or figures denote known or definite numbers; and mathematical operations or calculations, employing the *numerals,* constitutes the science of *Arithmetic.*.

The letters of the alphabet may denote any numbers or values whatever; either known or unknown, and mathematical operations, or processes, employing *letters,* or a combination of numerals and letters, constitutes the science of *Algebra.*

The *method of calculation* is identical for both Arithmetic and Algebra, and many of the symbols adopted are common to both. In both these subjects the various symbols employed, represent numbers, their relation with one another, and the operations performed with them.

The use of symbols lessens mental labor and requires much less space in printing or writing.

Everyone is familiar with many symbols met with in daily experience.

Any *quantity* may be designated by a symbol, such as a letter of the English alphabet or a letter of the Greek* alphabet, or by any other conventional sign. The sign $ placed before the number 10 means a certain amount of *money;* $10, denotes ten dollars.

The flexibility of the system employing the Arabic numerals is wonderful. By properly arranging these 10 symbols, the largest and the smallest conceivable number may be expressed.

The following arrangements of the figure 1 and the symbol 0 are important.

---

* See page 7 for Greek alphabet.

10	denotes	*ten.*
100	"	*one hundred.*
1,000	"	*one thousand.*
10,000	"	*ten thousand.*
100,000	"	*one hundred thousand.*
1,000,000	"	*one million.*
10,000,000	"	*ten million.*
100,000,000	"	*one hundred million.*
1,000,000,000	"	*one billion;* or one thousand million.

The use of the *decimal point* (see page 26) with this system still further adds to simplicity of expression.

Example 1: Suppose a collection of Arabic numerals denotes a number as: 137896;
which translated into common language is, one hundred and thirty-seven thousand eight hundred and ninety-six. The figures present a brief expression; yet everyone who reads can understand its meaning. Note the time and the ink, as well as the physical and mental exertion saved by employing the *symbols* or the figures.

The symbols as arranged, should properly be put into English as follows:

One hundred thousand, and thirty thousand, and seven thousand, and eight hundred, and ninety, and six.

As an arithmetical process the arrangement would indicate:

$$\begin{array}{r} 100000 \\ 30000 \\ 7000 \\ 800 \\ 90 \\ 6 \\ \hline 137896 \end{array}$$

This arrangement will be referred to later under *exponents,* page 32.

To denote a number *less than* 1, the Arabic numerals are arranged one over the other and the arrangement is called a *fraction.*

$\frac{1}{2}$ denotes one-half of one; which is the *reciprocal* of two.

$\frac{3}{2}$ denotes one-half of three; or three halves of one; or simply three halves.

$\frac{1}{3}$ denotes one-third of one.

$\frac{1}{10}$ denotes one-tenth of one.

$\frac{10}{3}$ denotes one-third of ten; or ten-thirds of one; or simply ten-thirds.

$\frac{1}{137896}$ therefore denotes a comparatively small number; as compared with 1.

Suppose a decimal point is placed between the 7 and the 8 as: 137.896.

Then the meaning is: one hundred and thirty-seven, and eight hundred and ninety-six *one thousandths;* which may be expressed in symbols as:

$$137\tfrac{896}{1000};$$

$$\text{or as:}\quad \tfrac{137896}{1000}=(\tfrac{137000}{1000}+\tfrac{896}{1000}).$$

Other examples are:

$$1000.001=1000\tfrac{1}{1000}=\tfrac{1000001}{1000};$$
$$100.01\ =100\tfrac{1}{100}=\tfrac{10001}{100};$$
$$10.1\ =10\tfrac{1}{10}=\tfrac{101}{10}.$$

PROBLEM 1: Express the following in the blank spaces using Arabic numerals or characters:

Seven thousand and twenty-one.

Answer.

PROBLEM 1a: One million, twenty-seven thousand, and forty-two.

Answer.

PROBLEM 1b: Sixty-six million, sixty thousand, and sixty.

Answer.

PROBLEM 1c: Seven hundred ninety-six, and twelve one hundredths. (decimal.)

Answer.

PROBLEM 1d: Seven hundred ninety-six, and twelve thousandths. (decimal.)

Answer.

PROBLEM 1e: Express the following in the blank spaces, using Arabic numerals:

Three-fourths.	Eight-ninths.	Two-tenths.
Answer	**Answer**	**Answer**

PROBLEM 1f: Fourteen one-hundredths. Twenty-seven one-thousandths.

Answer. **Answer**

LESSON II

USE OF LETTERS IN MATHEMATICS

The use of the letters of the English alphabet in mathematical processes will be explained more fully later on, and will be used very extensively in all the various subjects of the different courses. Suffice it to say if the following numerical values be assigned to the given letters.

$a=1$
$b=2$
$c=3$
$d=4$
$e=5$
$f=6$
$g=7$
$h=8$
$i=9$

then 137896 *could* be expressed by *acghif;* but unfortunately *when letters* are used to *denote numbers* their **arrangement** has a very different meaning attached to it, than has the corresponding arrangement of numerals. According to the adopted method of expressing a number by the use of letters in algebraical operations,

acghif with the numerical value assigned to each letter as above, would indicate:

$$282{,}480; \ (a\times c\times g\times h\times i\times f)$$

being the *product* of the numerals denoted by the individual letters.

That is, in *Algebra, bc* means *b multiplied by c,* and substituting the numerical equivalents assumed above, *bc* would equal 2 times 3, or 6.

Compare with the arrangement of numerals as explained on page 20.

The Roman Numerals I, II, III, IV, V, VI, VII, VIII, IX, X, corresponding to the Arabic numerals on page 10, cannot be used conveniently in mathematical processes.

Expressing 1918 in Roman numerals as

MDCCCCXVIII

will serve to show how cumbersome is the mere expression of a number, using this system, not to mention the complexity of multiplying the date as expressed in the Roman system, by IV. In the Roman system in addition to the equivalents already given, M denotes 1000; D denotes 500; C denotes 100; and L denotes 50. To express 600 it will be necessary to write DC.

The following is a statement recently issued:

> **Washington, Oct. 9.** Secretary McAdoo today instructed the supervising architect of the treasury to use Arabic instead of Roman numerals on all public buildings.
>
> The order was issued because of the difficulties the average citizen finds in quickly interpreting Roman numerals.

Example 2. Using the Roman numerals as characters, the number 2241 would be expressed by

MMCCXLI

Attention may be directed to the *position* of the characters relative to each other as determining the value of a number.

For example: IV means 4, while VI means 6.
IX means 9, while XI means 11.
XL means 40, while LX means 60.

A table showing the comparative arrangement of some of the Arabic and Roman numerals is given below:

1, I.	21, XXI.	41, XLI.	61, LXI.
2, II.	22, XXII.	42, XLII.	62, LXII.
3, III.	23, XXIII.	43, XLIII.	63, LXIII.
4, IIII or IV.	24, XXIV.	44, XLIV.	64, LXIV.
5, V.	25, XXV.	45, XLV.	65, LXV.
6, VI.	26, XXVI.	46, XLVI.	66, LXVI.
7, VII.	27, XXVII.	47. XLVII.	67, LXVII.
8, VIII,	28, XXVIII.	48, XLVIII.	68, LXVIII.
9, IX.	29, XXIX.	49, XLIX.	69, LXIX.
10, X.	30, XXX.	50, L.	70, LXX.
11, XI.	31, XXXI.	51, LI.	71, LXXI.
12, XII.	32, XXXII.	52, LII.	72, LXXII.
13, XIII.	33, XXXIII.	53, LIII.	73, LXXIII.
14, XIV.	34, XXXIV.	54, LIV.	74, LXXIV.
15, XV.	35, XXXV.	55, LV.	75, LXXV.
16, XVI.	36, XXXVI.	56, LVI.	76, LXXVI.
17, XVII.	37, XXXVII.	57, LVII.	77, LXXVII.
18, XVIII.	38, XXXVIII.	58, LVIII.	78, LXXVIII.
19, XIX.	39, XXXIX.	59, LIX.	79, LXXIX.
20, XX.	40, XL.	60, LX.	80, LXXX.

90, XC. 100, C. 200, CC. 300, CCC. 400, CCCC.
500, D. 1000, M. 2000, MM.

PROBLEM 2: Express the following numbers in the blank spaces, using Roman numerals.

450. Two hundred and twenty.

Answer. Answer.

PROBLEM 2a:

800. Nine thousand nine hundred and ninety-nine,

Answer. Answer.

PROBLEM 2b:

2582.

Answer.

PROBLEM 2c: Using Roman Numerals, express

1885. 1915.

Answer. Answer.

LESSON III

ALGEBRAIC SYMBOLS

The symbols employed in Arithmetic and in Algebra may be separated into four classes as follows:

1. Symbols denoting Quantity.
2. Symbols denoting Relation.
3. Symbols denoting Operation.
4. Symbols denoting Abbreviation.

The symbols denoting quantity and operation have already been discussed to some extent. The other symbols will be considered as follows:

Any arrangement of letters with algebraic symbols is called an *Algebraic Expression.*

Thus $a+b$; $5a-2b$; $az+bx-cy$ are all algebraic expressions.

Numerical expressions might be; $5+17=22$; $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$: and $86-5-19=62$, in which no *letters* are used.

THE SIGN OF EQUALITY, $=$ is expressed in words as *"equals"* or *"is equal to"*. $a=c$ denotes that a number denoted by a, is equal to another number denoted by c. The *expression* denoting the equality of two or more numbers or two or more quantities is called an *"equation"*. Thus $x-y=36$, is an equation; as is also $a=c$, and $xy=4p$.

The SIGNS OF INEQUALITY, $>$ and $<$ are sometimes employed. $>$ means *"is greater than"* and $<$ means *"is less-than"*. The symbol of inequality always points towards the *smaller* quantity.

For example: $a > b$ means, a is *greater than b*.

while $a < b$ means, a is *less than b*.

In a given problem, it often happens that numbers occupy corresponding *relations,* while differing in *value.* In such cases, a may denote one value, a' another, and a'' another. These are read:

a'; *a prime.*
a''; *a two prime;* or *a double prime.*
a'''; *a three prime.*

Sometimes letters may be used with *subscripts,* as:

a_1; read *"a sub one".*
a_2; read *"a sub two".*
a_3; read *"a sub three".*

PROBLEM 3: Express the following in the form of a numerical expression, observing the proper consecutive order of arrangement. An individual received 250 dollars; spent 75 dollars; earned 15 dollars; lost 17 dollars. How much money has he to his credit?

Answer.

PROBLEM 3a: Express the following as an algebraic *expression;* To a is added c, and subtracted b; giving as a result, b, less a, less c.

Answer.

PROBLEM 3b: Express the relation, using the proper *symbols,* that 16 is equal to a.

Answer.

PROBLEM 3c: That 16 is greater than b.

Answer.

PROBLEM 3d: That 4 is less than 5.

Answer.

PROBLEM 3e: That x is less than 25.

Answer.

PROBLEM 3f: That 481 is less than y.

Answer.

LESSON IV

SYMBOLS OF OPERATION

In almost all mathematical processes, whether employing figures or letters, time may be saved by adopting certain symbols denoting *"operation"*.

Some of the common symbols denoting mathematical processes or operations are as follows:

THE SIGN OF ADDITION:

$$+$$

Read *"plus"*.

$25+7$ means that to twenty-five is added seven, making their *sum* thirty-two.

$a+b$ means that a number denoted by a, is to be added to another number denoted by b. If a denotes, or stands for 25, while b denotes 7, then $a+b=32$. $a+b+c$ means that a number denoted by a is to be added to a number denoted by b and their sum is to be added to a third number denoted by c.

When numbers are added together the resulting number is called their *sum*.

The *order* in which numbers are added, will make no difference in the *numerical* value of their sum.

THE SIGN OF SUBTRACTION:

$$-$$

Read *"minus"*.

$25-7$ means that from the number twenty-five is taken the number 7, making their *difference* eighteen.

$a-b$ means that from a number denoted by a is taken or subtracted a number denoted by b. If a denotes 25 and b denotes 7 then $a-b=18$.

The number from which another number is taken is called the **minuend**; while the number taken away is called the **subtrahend**.

THE SIGN OF MULTIPLICATION:

$$\times$$

Read *"times"* or *"multiplied by"*.

25×7 means that 25 is taken seven times; or is multiplied by seven.

Multiplication is simply **a short method of** performing **addition.** If the number 25 is written down seven times, and all seven of the twenty-fives added together, the result is:

$$\begin{array}{r} 25 \\ 25 \\ 25 \\ 25 \\ 25 \\ 25 \\ 25 \\ \hline 175 \end{array}$$

The same result would have been obtained had *seven* been added *twenty-five times.*

The addition of the seven twenty-fives is further illustrated by:

$$\begin{array}{r} 25 \\ 25 \\ 25 \\ 25 \\ 25 \\ 25 \\ 25 \\ \hline 35 \\ 140 \\ \hline 175 \end{array}$$

Seven fives are first added together, giving 35; next seven twenties are added together, giving 140, and the two sums added, giving 175.

Expressing the operation as one of multiplication gives:

$$\begin{array}{r} 25 \\ 7 \\ \hline 35 \\ 140 \\ \hline 175 \end{array}$$

The result obtained by multiplying two or more numbers together is called the *product,* while the numbers which are multiplied together are called *"factors"* of the product. $a \times b$ means that a number denoted by a is to be multiplied by a number denoted by b. If a denotes 25; while b denotes 7, $a \times b = 175$.

$a \times b \times c$ indicates that the product of a multiplied by b is to be multiplied by c. a, b and c are called the *factors* of the result.

Suppose a denotes 25, b denotes 5, and c denotes 2; then $a \times b \times c = 25 \times 7 \times 2 = 350$.

In the operation of multiplying *two* numbers, the number to be multiplied is called the *"multiplicand"*, while the number by which the multiplicand is multiplied is called the *"multiplier"*.

As indicated above, 25 is the multiplicand and 7 is the multiplier. It is true that 7 might be the multiplicand and 25 the multiplier; but it is common practice to call the larger number the multiplicand, and the smaller number the multiplier.

When letters are used to denote numbers, as in Algebra, the sign of multiplication is usually omitted; thus abc means the same as $a \times b \times c$.

When expressing a number by using the Arabic numerals, it is the sign of addition that is omitted; the sign of multiplication is never omitted in *arithmetical* expression.

A *point* is used by some writers to denote multiplication instead of the sign $\times$. In such cases $4 \cdot 5 : 3$ means $4 \times 5 \times 3$. This is not a practice to be encouraged. The word *"of"* is sometimes used instead of the multiplication sign $\times$, to denote multiplication. This word *"of"* is used very often when multiplying fractions. For example; instead of writing $\frac{5}{9} \times 90$, the expression $\frac{5}{9}$ of 90 is used.

The saving effected by the common method of multiplication is apparent when dealing with *large* numbers. Suppose the problem is given to multiply **5280** by **123**. By addition it would be necessary to write down 5280 *one hundred and twenty-three times,* forming a column, and then adding. By multiplication:

$$\begin{array}{r} \mathbf{5280} \\ \mathbf{123} \\ \hline 15840 \\ 10560 \\ 5280 \\ \hline 649440 \end{array}$$

The *partial* products are shown as follows:

$$
\begin{array}{r}
5280 \\
123 \\
\hline
\end{array}
$$

$$
\begin{array}{rl}
\left.\begin{array}{r}
240=80\times 3 \\
600=200\times 3 \\
15000=5000\times 3
\end{array}\right\} & 3\times 5280=\ 15840 \\
\left.\begin{array}{r}
1600=80\times 20 \\
4000=200\times 20 \\
100000=5000\times 20
\end{array}\right\} & 20\times 5280=105600 \\
\left.\begin{array}{r}
8000=80\times 100 \\
20000=200\times 100 \\
500000=5000\times 100
\end{array}\right\} & 100\times 5280=528000 \\
\hline
649440 & \qquad\qquad 649440
\end{array}
$$

The **Italian Short Proof Method** of Checking Multiplication. It is sometimes of value to be able to quickly "check" the result of multiplying two large numbers by each other.

One method is called the *"Italian Short Proof Method"* and is explained as follows:

Suppose 172856 is to be multiplied by 375, expressed as;

A...........172856	Add digits giving 29.	Divide 29 by 9 giving 3 and *remainder* of 2.
B.................375	Add digits giving 15.	Divide 15 by 9 giving 1 and a *remainder* of 6.
864280		
1209992		
518568		
C........64821000	Add digits giving 21.	Divide 21 by 9 giving 2 and a *remainder* of 3.

Proof. In line A, add together all the digits, divide the sum by 9 and place the *remainder* in the square designated by 1, figure 1.

In line B, add together all the digits, divide the sum by 9, and place the *remainder* in the square designated as 2 in figure 1.

1 **2**	3 **3**
6 2	**3** 4

Fig. 1

Multiply the number in square 1 by the number in square 2, divide the product by 9 and place the *remainder* in square designated as 3. In this case $2\times 6=12$ and 12 divided by $9=1$; with a *remainder* of 3.

In line C, add together all the digits, divide the sum by 9, and if the multiplication has been properly performed, the *remainder* will be the same as the number placed in square designated by 3.

Place the *remainder* in square designated by 4. It happened in this particular case that the *remainder* number in square 3, was 3.

American Short Proof Method of Checking Multiplication. Another method of proving the result or product of multiplying two numbers together, which might be called the *"American Short Proof Method,"* is as follows:

Suppose 24532 is to be multiplied by 13685.

```
A..............24532 Adding digits gives 16; adding again; 1+6=7
B..............13685 Adding digits gives 23; adding again; 2+3=5
               -----
               122660                   Multiplying 7 by 5=35
              196256                          Adding 3 and 5=8
             147192
            73596
           24532               D-------------------E
         ---------
C..........335720420 Adding digits gives 26; adding again gives 8.
```

Proof. Add the digits in the multiplicand; designated by **A.** In this case giving the number 16. Again add digits giving 7.

Add the digits in the multiplier **B,** giving in this case 23. Again add digits, giving 5. Multiply 7 by 5 giving 35 and adding digits 3+5 giving 8.

Add all the digits in the product **C,** giving in this case 26. Again add digits 2+6 giving 8.

Always add digits until a number containing only *one* figure results.

If the result of the *operations* indicated above the line **D E** is the same as the result of the operations indicated below the line, the multiplication has been performed correctly.

PROBLEM 4: *Express* the following, using Arabic numerals and symbols. Fourteen, plus ten, minus eight, multiplied by sixty.

Answer.

PROBLEM 4a: Five times five, minus twenty, times two, plus ten, minus twenty.

Answer.

PROBLEM 4b: Write the *numerical result* of the following: Ten times four, minus fifteen, plus five, times five, minus one hundred and fifty.

Answer.

PROBLEM 4c: Minus twenty-five, plus one hundred and fifty, times two, times twenty-two, plus twenty-five.

Answer.

PROBLEM 4d: Multiply 155 by 5, and check the product by addition. That is, prove by addition that the result is correct.

Answer.

PROBLEM 4e: Multiply 15 by itself and check the result by the process of addition.

Answer.

PROBLEM 4f: Multiply 999999 by 888 and check the result by the Italian short proof method.

Answer.

PROBLEM 4g: Multiply 123456789 by 4321 and check the result by both the Italian and the American short proof methods.

Answer.

LESSON V

THE SIGN OF DIVISION:

$$\div$$

Read *"divided by."*

$25 \div 5$ indicates that 25 is to be divided into 5 equal parts. This operation may be also expressed by $\frac{25}{5}$

Applying to Algebra, $a \div b$ indicates that a number denoted by a is to be divided by a number denoted by b. This may also be expressed by $\frac{a}{b}$ which does away with the *symbol* of division.

Division is a short method of performing subtraction. (Note the relation between multiplication and division.)

For example: suppose 1492 is divided by 2, giving 746; 1492 minus 746=746. Again suppose from the number 25 is subtracted the number 5, giving 20; from which is subtracted 5 giving 15; from which is subtracted 5 giving 10, and from which is subtracted 5 giving as a result 5. This operation proves that 25 divided by 5 is equal to 5.

It is thus evident that *division is* an exact *reverse* operation *from* that of *multiplication.*

When a given number is divided by another number, the given number is called the *dividend,* the other number is called the *divisor,* while the result is called the *quotient.* Proof of the correctness of the result of division: *Multiply* the *quotient by* the *divisor,* which should give as a product the same numerical value as that of the dividend.

Example. Divide 175 by 7.

$$\begin{array}{r}
25 \\
\hline
7|175 \\
14 \\
\hline
35 \\
35 \\
\hline
\end{array}$$

To prove that the quotient is correct, multiply 25 by 7 which gives as a product, 175.

PROBLEM 5: Divide 649,440 by 123, and prove that the result is correct.

Answer.

PROBLEM 5a: Divide 11190 by 746, and prove the result.

Answer.

PROBLEM 5b: If the divisor is 746 and the quotient is 30, find the dividend. Prove result.

Answer.

LESSON VI

DECIMALS

The word decimal is derived from the Latin word *decimus,* meaning *tenth.*

A *decimal system* would be a system based upon *tenths.* The *Metric* system is such a system.

To facilitate expression, a point or period is employed to denote the decimal relation of numbers.

For example:

$\frac{1}{10}$ may be expressed **0.1.**

$\frac{1}{100}$ by **.01.**

$\frac{32}{1000}$ by **.032.**

20$\frac{4}{50}$ by **20.08.**

The *decimal point,* to a certain extent replaces the common fraction. Some fractions cannot be expressed in absolute value, as a decimal, as may be demonstrated by such a fraction as $\frac{1}{3}$, which can only be *approximately* expressed as a decimal as 0.33333 . . . . . (See page 38 for explanation of the sign of *continuation.*) This may be illustrated as follows:

$\frac{1}{3}$ (one-third) = 1 divided by 3.

```
    .3333  etc.
 3)1.0000
    9
   ----
    10
     9
   ----
     10
      9
   ----
      10
       9
   ----
```

The table on the next page gives the decimal equivalents for a few of the common fractions.

TABLE OF DECIMAL EQUIVALENTS OF COMMON FRACTIONS

$\frac{1}{64}$	$= .015625$	$\frac{5}{16}$ or $\frac{10}{32}$ or $\frac{20}{64}$	$= .3125$
$\frac{1}{32}$ or $\frac{2}{64}$	$= .03125$	$\frac{3}{8}$ or $\frac{12}{32}$ or $\frac{24}{64}$	$= .375$
$\frac{3}{64}$	$= .046875$	$\frac{7}{16}$ or $\frac{14}{32}$ or $\frac{28}{63}$	$= .4375$
$\frac{1}{16}$ or $\frac{2}{32}$	$= .0625$	$\frac{1}{2}$ or $\frac{16}{32}$ or $\frac{32}{64}$	$= .5$
$\frac{3}{32}$	$= .09375$	$\frac{9}{16}$ or $\frac{18}{32}$ or $\frac{36}{64}$	$= .5625$
$\frac{1}{8}$ or $\frac{4}{32}$ or $\frac{8}{64}$	$= .125$	$\frac{5}{8}$ or $\frac{10}{16}$ or $\frac{20}{32}$	$= .625$
$\frac{9}{64}$	$= .140625$	$\frac{11}{16}$	$= .6875$
$\frac{5}{32}$ or $\frac{10}{64}$	$= .15625$	$\frac{3}{4}$	$= .75$
$\frac{11}{64}$	$= .171875$	$\frac{13}{16}$	$= .8125$
$\frac{3}{16}$ or $\frac{6}{32}$ or $\frac{12}{64}$	$= .1875$	$\frac{7}{8}$	$= .875$
$\frac{7}{32}$ or $\frac{14}{64}$	$= .21875$	$\frac{15}{16}$	$= .9375$
$\frac{1}{4}$ or $\frac{8}{32}$ or $\frac{16}{64}$	$= .25$		

The application and use of the decimal point may be comprehended by examples as follows:

$0.1 = \frac{1}{10}$ *"one-tenth"* (equal to ten "one hundredths")

$.01 = \frac{1}{100}$ *"one hundredth"* (equal to ten "one thousandths")

$.001 = \frac{1}{1000}$ *"one thousandth"* (equal to ten "ten thousandths")

$.0001 = \frac{1}{10000}$ *"one ten thousandth"* (equal to ten "hundred thousandths"

$.00001 = \frac{1}{100000}$ *"one hundred thousandth"* (equal to ten "millionths")

$.000001 = \frac{1}{1000000}$ *"one millionth"*

Applying the Arabic numeral system, it is evident that:

0.2=two-tenths; 0.3=three tenths; 0.4=four-tenths; etc.

.02=two hundredths; .03=three hundredths; etc.

.002=two thousandths; .003=three thousandths; etc.

Next combining tenths and hundredths:

0.23=twenty-three hundredths; that is two-tenths, being equal to twenty *"one hundredths"* added to three hundredths, gives twenty-three *"hundredths"*.

Likewise; 0.235 means two hundred and thirty-five, "thousandths"; since *"two-tenths"* equals two hundred *"one thousandths"*, and *"three one hundredths"* equals thirty *"thousandths"*.

It is evident from the foregoing explanation that decimals have to do with tenths and *submultiples* of tenths.

The multiplication of decimals by numbers that are not fractions, called *"integers"* or "whole numbers", is important and several examples will be given.

Example 6: Multiply six hundred and twenty-five ten thousandths, by eight.

$$\begin{array}{r} .0625 \\ 8 \\ \hline .5000 \end{array}$$

Example 6a: Multiply fifteen thousand six hundred and twenty-five millionths by thirty-six.

$$\begin{array}{r} .015625 \\ 36 \\ \hline 93750 \\ 46875 \\ \hline .562500 \end{array}$$

Compare the decimal values of $\frac{1}{64}$ and $\frac{9}{16}$ on page 27. The following example shows how to multiply one decimal by another decimal.

Example 6b: Multiply one hundred and twenty-five thousandths, by twenty-five hundredths.

$$\begin{array}{r} .125 \\ .25 \\ \hline 625 \\ 250 \\ \hline .03125 \end{array}$$

Rule for "pointing off" or for locating the position of the decimal point in the product of two decimals.

Beginning at the **right** *of the product "point off" as many figures* (digits) *as there are decimal places in both multiplicand and multiplier.* In the last example there are *three* decimal places in the multiplicand, and *two* places in the multiplier: a total of *five* places. So beginning at the right hand figure (which is 5) of the product, count

to the *left* five decimal places, and place the point to the left of the fifth figure. In this case there were only four figures, so a *cipher must be added* as indicated. Had there been only three figures, then two ciphers would have been added.

In this case compare the multiplication of the decimal equivalent of ⅛ and ¼, page 27.

$\frac{1}{8}\times\frac{1}{4}=\frac{1}{32}$. $\frac{1}{32}=$

$$
\begin{array}{r}
.03125 \\
32\overline{)1.00000} \\
96 \\
\hline
40 \\
32 \\
\hline
80 \\
64 \\
\hline
160 \\
160
\end{array}
$$

The multiplication of decimals may be proved, the same as the multiplication of other numbers, by dividing the product, by the multiplier, which should give the multiplicand.

The division of a decimal number by an integer has already been shown, on pages 26 and 29. The proper location of the decimal point is evident from these examples.

The division of one decimal number by another decimal number is illustrated by:

Example 6c: Divide three thousand one hundred and twenty-five, hundred thousandths, by one thousand eight hundred and seventy-five, ten thousandths.

$$
\begin{array}{r|l}
 & 166\ldots. \\
\hline
.1875).0312 & 500 \\
187 & 5 \\
\hline
125 & 00 \\
112 & 50 \\
\hline
12 & 500 \\
11 & 250 \\
\hline
\end{array}
$$

The quotient in this case is an example of a so-called **"repeating decimal"** a number that cannot be expressed by a proper or a common fraction.

The further the process of division is carried out, the more sixes will appear in the quotient.

It is interesting to *"prove"* such an example:

$$\begin{array}{r}
.1875 \\
.166 \\
\hline
11250 \\
11250 \\
1875 \\
\hline
.0311250
\end{array}$$

The product differs from the dividend after the first two figures.

Had the division been carried to .16666 and the proof applied, the product would have been more nearly like the dividend. It would require an *"infinite"* number of sixes to prove exactly.

PROBLEM 6: Express as a decimal number, fifteen thousand six hundred and twenty-five, hundred thousandths.

Answer.

PROBLEM 6a: Express as a decimal, $\frac{1}{250}$, and express the answer in words.

Answer.

PROBLEM 6b: Multiply .004 by 1000.

Answer.

PROBLEM 6c: Divide 0.625 by .015625.

Answer.

PROBLEM 6d: Express 0.328125 as a common fraction.

Answer.

Note.

As a result of extended experience, it may be stated that college students make more errors in placing the decimal point than in any other operation. Students should exercise the utmost care in all mathematical processes involving decimals.

PROBLEM 6e: Express 18⅛ as a fraction.

Answer.

LESSON VII

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS.

The rule for ascertaining the "*sign*" to prefix to the product of two numbers is very simple, but should be firmly fixed in mind, since it applies in both Arithmetic and in Algebra.

The *product obtained by multiplying two numbers with unlike signs before them should be preceded by a* **minus** *sign.* If the two numbers are preceded by *like* signs, the sign of the product is *plus.*

Example 7: Multiply +55 by —25.

$$\begin{array}{r} +55 \\ -25 \\ \hline 275 \\ 110 \\ \hline -1375 \end{array}$$

Multiply —55 by —25.

$$\begin{array}{r} -55 \\ -25 \\ \hline 275 \\ 110 \\ \hline +1375 \end{array}$$

The same rule applies to algebraical multiplications; a multiplied by $-b = -ab$; $-a$ multiplied by $-b = ab$.

EXPONENTS.

If a number is multiplied by *itself* any number of times, the result or *product* is called a "*power*" of that number.

If 25 be multiplied by itself, the result, 625, is said to be the second power of 25. If 25 be multiplied by itself, and the product again by 25, the result, $(625 \times 25) = 15625$, is called the *third power* of 25.

The third power of 3 is 27.

The sixth power of 2 is 64.

The third power of 10 is 1000.

In order to indicate to what power any number is raised, a small figure is placed to the right and slightly above the number, to indicate the power; or to indicate how many times the number is to be multiplied.

This small figure is called an *"exponent"* or *index,* and may be a negative as well as a positive number, and may also be a fraction.

For example:

$$5^2=5\times 5=25.$$
$$5^3=5\times 5\times 5=125.$$
$$4^2=4\times 4=16.$$
$$4^3=4\times 4\times 4=64.$$
$$3^2=3\times 3=9.$$
$$3^3=3\times 3\times 3=27.$$

The powers of **10** are of considerable importance in engineering calculations and some explanations will be made regarding them.

$10^1=10$*, read simply 10; or ten to the first *power.*

$10^2=10\times 10=100$, read "ten *squared*".

$10^3=10\times 10\times 10=1000$, read "ten *cubed*", or ten to the "third power".

$10^4=10\times 10\times 10\times 10=10000$, read *ten to the "fourth power".*

$10^6=1{,}000{,}000$ (one million), read *ten to the "sixth power".*

$10^8=100{,}000{,}000$ (one hundred million), read *ten to the "eighth power".*

$10^9=1{,}000{,}000{,}000$ (one thousand million or one billion), read *ten to the "ninth power".*

Suppose a given number is 42,000,000; by adopting the *"power of ten"* method of expressing the number, it would be $\mathbf{4.2\times 10^7}$.

It saves space and ink to write 10 with its exponent instead of printing or writing all the figures denoting the larger numbers, hence the following rule regarding the expressions for powers of ten.

Rule. *In order to write out the number that is expressed by* 10 *with any exponent, add as many zeros to* 1 *as indicated by figure of the exponent.*

This rule is illustrated by the powers of 10 indicated above; 10^9 is expressed by 1 with nine ciphers after it.

* When no exponent is written, 1 is always understood.

In Algebra the exponent has the same meaning as when used in *Arithmetic.*

For example:

a^2 means "*a squared*" or "*a* to the second power", same as $a \times a$.

a^3 means "*a cubed*", or "*a* to the third power", same as $a \times a \times a$.

a^1 is the same as *a*, "*a* to the first power".

Rule. *When multiplying two or more similar numbers by each other, each having exponents,* **add** *their exponents together.*

For example:

$10^2 \times 10^4 \times 10^5 = 10^{11} = 100,000,000,000$: or one hundred billions.

Applying the rule to literal factors:

$a^2 \times a^5 \times a^1 = a^8$.

Suppose it is given that 4.2×10^7 is to be multiplied by 5×10^4; then $5 \times 4.2 \times 10^7 \times 10^4$ is the arrangement, and the result will be 21.0×10^{11} or 2.1×10^{12}. Considerable time is saved in the operation of multiplying, by adopting the *power of* 10 expression.

Rule. *When dividing one number by another similar number, but each having different exponents,* **subtract** *exponents.*

Example 7. Divide 25^3 by 25^{-2}.

Subtracting exponents gives $+3 - (-2) = 5$.

Therefore $25^3 \times 25^{+2} = 25^5 = 9,765,625$.

FRACTIONAL EXPONENTS. Exponents may be *fractional* as well as integers.

$4^{\frac{1}{2}}$ means four raised to the one-half power, and is often expressed by use of the so-called "*radical sign*" $\sqrt{\ }$.

Using this sign, $4^{\frac{1}{2}}$ is the same as $\sqrt{4}$. The one-half power of a number is also called the "*square root*" of the number.

The cube root of a number is expressed by the fractional exponent $\frac{1}{3}$, or by the sign $\sqrt[3]{\ }$.

A few numerical examples are given as follows:

$25^{\frac{1}{2}} = \sqrt{25} = \pm 5$; $\sqrt{625} = \pm 25$; $\sqrt[3]{125} = 5$; $\sqrt[3]{15625} = 25$.

PROBLEM 7: Multiply -25 by 25 and the result by -25.

Answer.

PROBLEM 7a: Multiply 5^3 by 4^2.

Answer.

PROBLEM 7b: Divide 5^5 by 5^3.

Answer.

PROBLEM 7c: Using the "*power of ten*" method, express the product of 1.5×10^5 by 9.2×10^2.

Answer.

LESSON VIII

NEGATIVE EXPONENTS

RECIPROCAL. The reciprocal of a number, is 1 divided by that number (Refer to page 10).

The reciprocal of 10 is $\frac{1}{10}$; of 25 is $\frac{1}{25}$; of x is $\frac{1}{x}$.

The reciprocal of 64, 32, 16, 8, 4 and 2 are shown as fractions in the table on page 27.

An application of reciprocals to powers may be alluded to here; since it becomes necessary at times to deal with numbers having *negative exponents.* Suppose a^{-1} is given in some mathematical process. The question naturally is asked how can a number be raised to a *negative* power?

Any number having a negative exponent is the same as the reciprocal of that number with a *positive* exponent having the same *numerical* value as the original negative exponent. That is:

$$5^{-3}=\frac{1}{5^3}\text{; or } a^{-1}=\frac{1}{a}\text{; or } a^{-5}=\frac{1}{a^5}$$

In proof of this, apply the rule for finding the exponent of the product of two or more powers of any number: (See page 34.)

Then $a^{-1}=a^{-2}\times a^1$.

Dividing both members of this equation by a^{-2} gives:

$\frac{a^{-1}}{a^{-2}}=\frac{a^{-2}\times a^1}{a^{-2}}$ or $\frac{a^{-1}}{a^{-2}}=a^1$ which is the same as $\frac{1}{a^{-1}}=a^1$; obtained by dividing both numerator and denominator by a^{-1}; or $\frac{1}{a^1}=a^{-1}$

This proof may be better understood after studying **Lesson XII,** page 53.

SYMBOLS DENOTING ABBREVIATION.

In mathematical processes or operations much time and labor is saved by employing certain signs or symbols denoting a long or complicated process or numerous short processes.

THE SIGNS OF AGGREGATION OR COLLECTION are,

the parentheses, ( );

the braces, $\}$ $\{$;

the brackets, [ ];

and the *vinculum,* ————————,

all of which indicate that the numbers included by them are to be considered *collectively,* as a unit.

For example:

$$(a+b)c;$$

$$\left\{ a+b \right\} c;$$

$$[a+b]c;$$

$$\overline{a+b} \times c,$$

all mean that the sum resulting from adding a to b is to be multiplied by c.

Each of the above four expressions may of course be written $ac+bc$.

Further examples of aggregation are:

$$(a+b)^2$$
$$[a+b]^2$$
$$\overline{a+b}^2$$

each of which denotes that the sum of a and b is to be multiplied by itself or is to be *squared.*

The following example shows the use of the signs of aggregation.

Example 8.

$$2b[8(a+b)^2-5a]=2b[8(a^2+2ab+b^2)-5a.]$$
$$2b[8a^2+16ab+8b^2-5a]=16a^2b+32ab^2+16b^3-10ab.$$

The student should be very careful indeed in all calculations where these signs of aggregation are employed. Very many mistakes are made by not observing the proper relations of numbers with which these symbols are used.

THE SIGN OF DEDUCTION:

$$\therefore$$

Read *"therefore"* or *"hence".*

Example:

$$ab = 10;$$
$$\therefore \ a = \tfrac{10}{b}$$

THE SIGN OF CONTINUATION;

$$\cdots\cdots$$

read *"and so on"*.

Example:

$$x, \quad x+y, \quad x+2y, \quad x+3y \;\ldots\ldots$$

read x, $x+y$, $x+2y$, $x+3y$ *"and so on"*.

Another example:

$$1+1+\frac{1}{1\times 2}+\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 3\times 4}\ldots\ldots$$

read "one plus one, plus one divided by 1 times 2, plus one divided by 1 times 2 times 3, plus 1 divided by 1 times 2 times 3 times 4, *and so on*".

See pages 26 and 29 for application of the sign of continuation.

PROBLEMS.

PROBLEM 8: Find the numerical value of 5^{-4}; 25^{-2}; 4^{-4}; $625^{-\frac{1}{2}}$; 1000000^{-6}.

Answer.

PROBLEM 8a: Find the numerical value of $(2+6)8$.

Answer.

PROBLEM 8b: Find the numerical value of $7[9(3+7)-125]$.

Answer.

PROBLEM 8c: Find the numerical value of $5[10(5+5)^2-1250]$.

Answer.

PROBLEM 8d: Find the numerical value of $\overline{5+5}^2+\overline{5-2}^3-49$.

Answer.

PROBLEM 8e. Find the numerical value of $[2(5+3)]^2$.

Answer.

LESSON IX

RELATING TO ALGEBRAIC EXPRESSIONS

AN ALGEBRAIC EXPRESSION, may be defined as the representation of a quantity by algebraic symbols; which may include both letters and figures. All of the following are algebraic expressions:

$$a; \quad 2a+3b+x; \quad x-\frac{1}{y}; \quad F=\frac{9}{5}C+32; \quad x^2-y^2-36=0.$$

(The last two expressions are *equations;* explained on page 53.)

A "TERM" is that part of an algebraic expression separated from the rest of the expression by the signs $+$ or $-$. The terms in the expression $x^2-y^2-36=0$ are x, $-y$, and -36.

The terms in $a^2+2ab+b^2=625$ are a^2, $2ab$, b^2 and 625.

A POSITIVE TERM is one that is *preceded* by a plus sign, as a^2, x^2, $2ab$, and 625. This is why the sign is often called the *positive* sign.

Whenever no sign precedes a term, $+$ is always understood.

A NEGATIVE TERM is one that is preceded by a minus sign, as $-y$, -36. For this reason the minus sign $-$, is often called the *negative sign.*

Great care should be observed *never to omit the minus sign before a negative term.* If no sign appears before a term, plus is understood.

A "MONOMIAL" is an algebraic expression consisting of only *one term;* as $4ax$; $-\frac{5}{8}x$; $-a$; $90mx$.

A "BINOMIAL" is an algebraic expression consisting of *two* terms; as $2x+2y$; $3x-a$; $2ax-3by$, and $-az+1$.

A "TRINOMIAL" is an algebraic expression consisting of *three* terms; as $2x+2y+z$; $-az+1-2ax$, and $a^2+2ab+b^2$.

A "POLYNOMIAL" is an algebraic expression consisting of more than one term, as x^2+y^2; $a^2+2ab+b^2$; $a^2+2ab+b^2=625$.

COEFFICIENTS. Any numerical or literal symbol, placed as a multiplier, before another numerical or literal symbol or combination of numerical and literal symbols, is called a coefficient. Greek letters are often used as literal coefficients to distinguish them readily from other symbols. As an example the expression for the area of a circle may be given; $A=\pi R^2$. The Greek letters more commonly used as coefficients in engineering literature and calculations, are β, γ, ρ, μ and

π. See page 7 for the Greek alphabet. The following letters of the English alphabet are also used considerably as coefficients: *a, b, c, d, k, l, m, n* and *p*.

Any figure or number may be employed as a coefficient, and may be either positive or negative in sign.

In the expression $4x^2+2xy+y^2=0$, the figures 4 and 2 may be considered as *"numerical"* coefficients; while in the expression $y^2=px$, p is the *"literal"* coefficient.

In the expression for the area of a circle in terms of the radius of the circle $\mathbf{A}=\pi\mathbf{R}^2$, **A** denotes the area in *square* units, while **R** denotes the radius in *linear* units of the same denomination. If **R** is expressed in inches then **A** is in *square inches;* while if **R** denotes feet, then **A** denotes *square feet.* The Greek letter π (pi) is a so-called *constant;* having the same numerical value regardless of the size of the circle.

ALGEBRAIC ADDITION AND SUBTRACTION.

A few examples will be given to explain more fully algebraical addition and subtraction.

Example 9: Add $2a$, $4b$, $3c$, $8b$ and $9a$.

Place like *terms* directly under each other and add, with proper regard for the signs of the terms, as follows:

$$\begin{array}{lll} 2a & 4b & 3c \\ 9a & 8b & \\ \hline 11a & 12b & 3c \end{array}$$

Answer is $11a \quad 12b \quad 3c$ This illustrates the addition of *"monomials"*.

Example 9a: Add $2x+2y$ and $2x-2y$.

$$\begin{array}{l} 2x+2y \\ 2x-2y \\ \hline 4x \end{array}$$

Answer is $4x$ This illustrates the addition of *"binomials"*.

Example 9b: Add $2xy+3ax-20y+10ax$ and $20y-10ax-3ac-2$.

$$\begin{array}{rrrrr} 2xy & -20y & +3ax & & \\ & 20y & +10ax & & \\ & & -10ax & -3ac & -2 \\ \hline 2xy & & +3ax & -3ac & -2 \end{array}$$

Answer is: $2xy \quad +3ax-3ac-2$

This illustrates the addition of *"polynomials"*.

An example illustrating algebraic *subtraction* will be given as follows:

Example 9c. Subtract $5xy-2x+y$ from $6xy-2x+y$.

The rule for algebraic subtraction is; *change all the signs of the subtrahend and proceed as in algebraic addition.* Hence the arrangement will be:

$$\begin{array}{r} 6xy-2x+y \\ -5xy+2x-y \\ \hline \end{array}$$

and adding gives: $xy+0+0$ or simply xy.

EVALUATING EXPRESSIONS.

The *numerical value* of any algebraic expression is the number obtained after substituting the numerical value of each symbol that has been ARBITRARILY assigned to it, and then performing all the operations indicated by the symbols of operation.

A few examples of finding the numerical value of algebraic expressions will be given as follows:

Let it be arbitrarily assumed that

a denotes 1.
b denotes 2.
c denotes 3.
x denotes 0.
y denotes 10.

from which assumptions find the numerical value of

$(a+b)\times(a+b)$. Making the proper substitutions,

gives: $(1+2)\times(1+2)$

$=3\times3=9$ **Answer.**

Evaluate:

$ax+by+c$

$(1\times0)+(2\times10)+3$

$0+20+3$

$=23$ **Answer.**

Evaluate:

$x^2+axy+y^2$

$0+(1\times0\times10)+10^2$

$=100$ **Answer.**

Any number multiplied by 0 (zero) is zero.

Assuming that:

$a=1$	$z=0$
$b=2$	$x=5$
$c=3$	$y=10$

Find the numerical value of the following algebraical expressions:

PROBLEM 9: $y^2=2ax$.

Answer.

PROBLEM 9a: $x^2+axy+y^2$.

Answer.

PROBLEM 9b: $x^2+y^2=a^2$.

Answer.

PROBLEM 9c: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Answer.

PROBLEM 9d: $3y^2=25x$.

Answer.

PROBLEM 9e: $5x^2=9y$.

Answer.

PROBLEM 9f: If $x=\frac{a}{2}$. Find the numerical value of C in $(C^{\frac{x}{a}}+C^{-\frac{x}{a}})$

Answer.

PROBLEM 9g: $y=x^{-b}$.

Answer.

The following problems illustrate algebraical addition:

PROBLEM 9h: Add $2x$; $3xy$; $-5z$; $8yz$; $-10x$; $3z$.

Answer.

PROBLEM 9i: Add $2cy-3az$; $2yc+2az$, and $-az+1$.

Answer.

PROBLEM 9j: Add $2x+3y-5a+5b$; $5a-5b-3y$; $5b-2x-1$, and $1-5b+a$.

Answer.

PROBLEM 9k: Subtract $3x+2$ from $2x+2$.

Answer.

PROBLEM 9l: Subtract $2ax+5cx^2-2by$ from $ax-by+7cx^2$.

Answer.

LESSON X

ROOTS

SPECIAL CASES OF ALGEBRAICAL AND ARITHMETICAL MULTIPLICATION

For the present consideration, suppose a denotes 20 and b denotes 5. Then $a+b=25$. The *square* of $25=25^2=25\times25=625$. The squaring of 25 is performed as follows:

$$\begin{array}{r} 25 \\ 25 \\ \hline 125 \\ 50 \\ \hline 625 \end{array} \qquad \text{or} \qquad \begin{array}{r} 25 \\ 25 \\ \hline 125 \\ 500 \\ \hline 625 \end{array}$$

The squaring of $(a+b)$ may be performed as follows:

$$\begin{array}{r} a+b \\ a+b \\ \hline ab+b^2 \\ a^2+ab \\ \hline a^2+2ab+b^2 \end{array}$$

The *letters* being multiplied by one another, as were the *figures* in the process of squaring 25.

Substituting the assumed numerical values of the letters, in the last algebraical product the following is true:

$$\begin{array}{r} a^2=400 \\ 2ab=200 \\ b^2=25 \\ \hline \end{array}$$

or $a^2+2ab+b^2 = 625$

The preceding operations illustrate a simple application of *"Algebra"*, to a mathematical process; showing the relation between an arithmetical process and an algebraical process. The arithmetical case is a specific one; while the algebraical case is a general one, when no numerical values are assigned to the letters. That is, the expression $a^2+2ab+b^2$ is a *typical* expression, which may denote a variety of numerical conditions.

For example, if $a=5$ and $b=5$, then $a^2+2ab+b^2=100$. Likewise $a^2+2ab+b^2$ might equal 144.

FACTORS AND ROOTS.

Considering the number **625**, it has been shown to be obtained by multiplying 25 by 25; it might also have been obtained by multiplying 125 by 5; or by $5\times5\times5\times5$ which shows that 625 is made up of the product of several numbers.

Those numbers which when multiplied together give a certain number, are called *"factors"* of the given number.

Two or more **equal factors** of any given number are called the **"roots"** of that number. (Refer to page 32 under *exponents.*) Each of two *equal* factors is called the *square* root.*

The number 125 is obtained from $5\times5\times5$; hence **5** is said to be the *"cube root"* of **125.** On the other hand, 125 is the *"cube"* of 5.

By following out the foregoing theory, the *"fourth root"* of 625 is 5.

It is interesting to note how rapidly *"cubes"* increase.
Take the case of 4, which is *twice* 2:—
The cube of **2** is $2\times2\times2=\mathbf{8}$.
The cube of **4** is $4\times4\times4=\mathbf{64}$.
The cube of 4 is *eight* times as great as the cube of 2.

In the algebraical case just considered, $a^2+2ab+b^2$ is the square of $(a+b)$, and $(a+b)$ is the square root of $a^2+2ab+b^2$.
The cube of $(a+b)$ may be obtained as follows:

$$
\begin{array}{r}
(a+b)\times(a+b)=a^2+2ab+b^2 \\
a+b \\
\hline
a^2b+2ab^2+b^3 \\
a^3+2a^2b+\ ab^2\quad\ \ \ \\
\hline
a^3+3a^2b+3ab^2+b^3.
\end{array}
$$

Then $(a+b)$ is the "cube root" of $a^3+3a^2b+3ab^2+b^3$.

When multiplying algebraical expressions by one another, always set *like terms* directly *under each other* for convenience in addition.

Note that in multiplying literal factors by each other their exponents are *added.* (algebraical addition.)

---

*The reason for the statement "*square*" root may be evident by considering a *square.* Multiplying together the numerical values of two sides of any square will give as a product the *area* of the square. If the side of any square is 4 feet, the area of the square is $4\times4=4^2=16$ *square feet.*

PROBLEM 10: Find the square of 2; 4; 6; 8 and 9.

Answer.

PROBLEM 10a: Find the cube of 2; 4; 6; 8 and 9.

Answer.

PROBLEM 10b: Calculate the square of $(x+y)$; $(2x+y)$; $(x+2y)$.

Answer.

PROBLEM 10c: Calculate the cube of $(2x+y)$; $(4+y)$; $(x+8)$.

Answer.

PROBLEM 10d: Find the algebraical expression for the *cube* of $a+b$.

Answer.

PROBLEM 10e: Find the algebraical expression for the *cube* of $a-b$.

Answer.

PROBLEM 10f: Write the algebraical expression for the result of $(x-y)^2+(z-x)^2$.

Answer.

LESSON XI

RATIO AND PROPORTION

A **ratio** expresses the *relative* magnitude of two quantities of the *same kind.*

A ratio is the relation of two values as expressed by *division.*

The ratio of 4 to 5 is expressed by $\frac{4}{5}$. This is simply a *fraction* whose numerical value is $4 \div 5$; which expressed as a decimal, is 0.8.

Therefore the ratio of $\frac{4}{5} = 0.8$. ($\frac{4}{5} \times \frac{2}{2} = \frac{8}{10}$); multiplying both numerator and denominator of the fraction $\frac{4}{5}$ by 2.)

Since it is not logical to compare quantities of different *kinds,* the result of dividing one quantity by another quantity of the *same kind,* but possibly different in amount, can be only a *number,* not denoting any particular *kind;* hence a ratio is sometimes alluded to as a *"pure number".* It might be called an *abstract number.*

It would not be logical or sensible to compare 3 horses with 7 houses; nor 3 apples with 7 automobiles, but it would be more consistent to compare 3 horses with 7 horses; 5 pounds with 10 pounds.

If 5 pounds be compared with 10 pounds and expressed as a ratio it will be:

$$\frac{5 \text{ pounds.}}{10 \text{ pounds.}}$$

The result is evidently not $\frac{5}{10}$ pounds; but simply $\frac{5}{10}$; or $\frac{1}{2}$; or 0.5; an *abstract* or *"pure number".*

The physical meaning in this case being that 10 pounds is twice as great as 5 pounds.

This consideration has a very important application in efficiency engineering, in comparing the useful output of any machine or device with the total input to the same machine or device.

The efficiency of any machine is the ratio of its useful output to the total input to the machine.

There are many *classes* of efficiency. One important efficiency met with in practice is **power** *efficiency.*

The **power efficiency of any machine** or device **is the ratio of the useful power output** from the machine or device, **to the total power input** to the same machine or device.

While there are many different kinds of efficiency, efficiency in general is always a ratio whose numerical value is always less than 1.

Percent efficiency is the *fractional value* multiplied by 100.

A ratio, then, *is* really *the number of times one quantity is contained in another* quantity *of the same kind.*

In Algebra, the ratio of **a** to **b** is expressed as **a : b** or as $\frac{a}{b}$, and **a** is called the *first "term"*, and **b** is called the *second "term"* of the ratio.

A **proportion** *is the expression of the* **equality** *of* **ratios.**

The following is an expression of a numerical proportion:

3 : 7 : : 9 : 21, and means that 3 is to 7 as 9 is to 21. The same relation may be expressed by:

3 : 7=9 : 21; or by $\frac{3}{7}=\frac{9}{21}$.

In any proportion the two middle figures are called the *"means"* while the two outer figures are called the *"extremes"*.

The symbol **:** means *"is* to", while **: :** means **"as"**, or *equals.*

It is well to note that the numerical value of a ratio may be **1**, more than **1**, or less than **1**.

PROBLEM 11: Find the numerical value of the ratio of 3,730 to 746.

Answer.

PROBLEM 11a: Find the ratio of the power output (power efficiency) to the power input for a motor whose output is 7460 watts (10 horse-power) and whose power input is 8288.8 watts. (11.1 horse-power.)

Answer.

PROBLEM 11b: Find the numerical value of the ratio of the area of a square to the area of a circle if the area of the square is 4 square inches and the area of the circle is 3.14 square inches.

Answer.

PROBLEM 11c: Given $a : 10 = 15 : 150$, find the numerical value of a.

Answer.

PROBLEM 11d: Given $\frac{a}{5} = \frac{10}{20}$, find the numerical value of a.

Answer.

LESSON XII

SOLUTION OF EQUATIONS

As the word *"equation"* implies, an equation is an expression of *equality* between two quantities. The expression may be one involving either numbers or letters, or both. That is, there are arithmetical equations and algebraic equations; as $2 \times 4 = 9 - 1$, and $2x = 5y$.

The study of the subject of engineering involving as it does, the consideration of equations of all kinds, it will be well to carefully consider a few important characteristics of equations and in particular the solutions of a variety of equations.

The following will be taken as an illustration of an equation containing both letters and figures; or numbers.

$$F = \tfrac{9}{5}C + 32.$$

The sign of equality is considered to separate any equation into two parts, the left-hand side and the right hand side as one faces the equation. The *sides* of an equation are sometimes alluded to as the left-hand *"member"* or the *first* member, and the right-hand *"member"*, or the *second* member.

The first rule to be observed in solving any equation is; when *a* TERM* *is changed from one side of the equation to the other* side the *algebraic* SIGN *of the term must also be changed.* If the sign of the term is + before the term is changed over, then the term must be preceded by a — sign after being changed. If — before change, it must be + after change. The principle underlying this rule may be illustrated by the following: given: $x - 5 = 2$ to find the numerical value of x.

Add 5 to both members of the equation giving:

$$x - 5 + 5 = 2 + 5, \qquad (-5 + 5 = 0)$$

whence $x = 7$. The same result is obtained by placing — 5 on the right hand side of the equation and changing the sign to +.

Given $y + 20 = 44$; to find the value of y.

Subtract 20 from both sides, giving:

$$y + 20 - 20 = 44 - 20;$$

whence $y = 24$.

Another rule is that *multiplying or dividing both sides of any equation by the same number, does not alter the value of the equation.*

---

* See page 40.

Also adding to or subtracting from both sides of any equation the *same number* does not affect the value of the equation,† and changing the sign of *all* the members of an equation does not change the value of the equation.

Suppose it is desired to solve the equation $\mathbf{F} = \frac{9}{5}\mathbf{C} + 32$ for **C**; the term containing C must be placed on the left-hand side of the equality sign, and F must be placed on the other side. Then the equation would be expressed:

$-\frac{9}{5}C = -F + 32$, which after changing *all the signs,* becomes:

$\frac{9}{5}C = F - 32$. As the equation now appears, $\frac{9}{5}C$ is the first or left-hand member, and $(F - 32)$ is the second or right-hand member. When solving any equation for a particular quantity, it is usual to divide both members of the equation by a number that shall make the *coefficient* of the particular quantity designated, equal to *unity.* In the above case both members of the equation must be divided by , giving:

$$C = \tfrac{5}{9}(F - 32).$$

Note the use of the parenthesis as explained on page 37. Perhaps the simplest form of an equation is when both sides are alike; as $y + x = x + y$; $2 + 4 = 4 + 2$. Such equations are called *"identical"* equations.

Another class of equations is very common, in which the two sides are equal, numerically, only upon some condition being imposed on some factor or term. An example is $\mathbf{2x = 10}$.

$2x = 10$ only upon condition that $x = 5$; then $2 \times 5 = 10$. If any other value is assumed for x, the two members would not be numerically equal to each other.

In such a case, x is called the *unknown* quantity.

That value of the unknown quantity in any equation, which makes the two members equal to each other, is called a *"root"* of the equation.

When the root of the equation is properly substituted in the equation it is said to *"satisfy"* the equation.

The process of finding the root or roots of an equation is called *solving* the equation.

Conditional equations are divided into classes or orders, according to their *degree,* or according the power of the unknown quantity.

The *"degree"* of an equation is the same as the highest degree or power of its unknown quantity.

† Equal quantities, equally affected, remain equal.

As examples:
$2x = a$; $cx = 10y$, and $x + y = 5$, are equations of the *first degree.*
$y^2 = 2b$, and $y^2 + 2x + b = 4$, are of the *second* degree.
Equations of the second degree are called *"quadratics"*.
$5x^3 = 10$, and $y^3 + 3y^2b + 5yb^2 + b^3$, are of the *third* degree.
Equations of the third degree are called cubical equations, or *cubics.* If the side of a square is multiplied by itself three times, the product will give the volume of a *cube.*

The exponents of the unknown quantities might be either positive, negative, fractional or integral, and each equation might have more than one unknown quantity.

PROBLEM 12: If **F** in the equation $\mathbf{F} = \frac{9}{5}\mathbf{C} + 32$, is equal to -40 find the numerical value of **C.**

Answer.

PROBLEM 12a: Given the equation $\mathbf{R} = \rho\frac{l}{\mathrm{A}}$, find the numerical value of **R** if $\rho = 10.8$, $l = 1$ and $\mathbf{A} = 4108.8$.

Answer.

PROBLEM 12b: Using the assumed numerical values; $x = 5$ and $y = 10$, find the effect of multiplying the equation $x^2 + x = 3y$, by 3.

Answer.

PROBLEM 12c: Given $\frac{z}{2} - 10 = 20$ to find the value of z.

Answer.

PROBLEM 12d: Given $8x + 4 = 10$ to find the value of x.

Answer.

PROBLEM 12e: Given $y = mx + b$; find the numerical value of m, when $y = 2$, $x = 4$ and $b = 5$.

Answer.

LESSON XIII

CONSTANTS AND VARIABLES

Any *quantity*, or any value *may* continue to *be constant* as time continues, *or it may change or vary.*

Any varying quantity is called a *variable;* while any value that continually remains the same is called a *constant.*

Of course a quantity may be constant for a certain interval of time and then may vary for a certain time. It is customary to state the conditions of variability when considering any particular problem, or during any mathematical discussion. A few examples of physical *constants* may be cited; as the force of gravity, denoted by g, which is constant for any given locality on the earth's surface; it has, of course, different values in different localities.

The diameter of the earth may be considered to be a constant.

The ratio of the circumference of any circle to its diameter is a constant; denoted by π and is taken as 3.1416 for ordinary calculations.

It is customary to denote constants or constant values, by using the first few letters of the alphabet, while the last few letters of the alphabet are used to denote variables.

Constants are denoted by ; a, b, c, d, g, h, k, m, and n. Variables are denoted by p, q, s, t, x, y, z.

Some of the letters of the Greek alphabet, as κ, π, and ω are used to denote constants and such letters as ϕ, θ, η are used to denote variables.

FUNCTIONS.

Whenever a quantity, denoted by y, depends upon another quantity, denoted by x, in such a manner that no change can be made in x without producing a corresponding change in y, then y is said to be a *function* of x.

The symbol $f\ (x)$ is used to denote a function of x, and is read the *"f function of x"*.

In a like manner $\phi\ (x, y)$ denotes a function of x and y, and is read *"the ϕ function* (phi function) *of x and y"*.

Further; the symbol

$y=f\ (x)$ is often used to express the fact that y is a function of x.

The fact that a change may be made in a quantity denoted by x or y, indicates at once that x and y denote variables. For brevity, x and y are therefore themselves called *variables.*

In any equation involving x and y, y is a function of x, if x is taken as the *independent* variable; while x would be a function of y if y be taken as the *independent* variable.

Example: $x^2-y^2-36=0$, is an equation involving the two variables x and y. One of these variables as x may be made the *independent* variable; then y will be the *dependent* variable. Suppose the above equation is solved for y, as follows:

$$-y^2=-x^2+36;$$
$$\text{or}\quad y^2=x^2-36$$
$$\text{and}\quad y=\pm\sqrt{x^2-36}$$

Now the function is expressed directly in terms of the independent variable x, and is said to be an *explicit* function, or y is said to be an explicit function of x.

If any numerical values are put in place of x in this last expression, *corresponding* numerical values may be obtained for y. One value will be assumed for x for the present. Let $x=10$; then $x^2=100$, and

$$y=\sqrt{100-36}=\sqrt{64}=\pm 8.$$

A large number of different values might be substituted for x and corresponding values for y found by the process of Arithmetic. This question will be considered more at length, later on.

Perhaps the meaning of the word *function* and the relation of functions may best be understood by considering a few concrete illustrations. It is well known that the area of a square is found by multiplying the values of two of its sides together. If **a** denotes one side of a square, then its area $=\mathbf{a}\times\mathbf{a}=\mathbf{a}^2$. The area of a square is a function of one of its sides. It makes no difference what the *size* of a square is, the above relation always holds. Suppose a side of a square *varies;* then its area varies as the *square* of its variable side. If y denotes the area of any square and x a side, then:

$$y=x^2$$

If x is doubled, y is increased *four times.*

PROBLEM 13: Given the equation $x^2+y^2=37$, find the numerical value of x, when $y=1$.

Answer.

PROBLEM 13a: Given the equation $x^2+10x+y^2-8y+41=36$, compute and properly tabulate the values of y corresponding with the following values of x; 0, 1, -1, 2, -2, 3, 4 and 5.

Answer.

PROBLEM 13b: Given the equation $\mathbf{F}=\frac{9}{5}\mathbf{C}+32$, compute and tabulate values of **F** corresponding with the following values of **C**; -40; 0; 32; 100; 1000.

Answer.

LESSON XIV

LOGARITHMS

The word logarithm is made up of the two Greek words (λδγορ) *logos,* meaning *"ratio"*, and (αριθρ) *arithmos,* meaning *"number"*.

The strict meaning of the word is, a *"ratio number"*.

Logarithms enable us to perform in a short time certain mathematical calculations that otherwise require considerable time and labor.

For example; multiplication and division, which are sometimes long and difficult operations, may be quickly and easily performed by methods involving logarithms and *"logarithmic tables"*.

Logarithms may also be employed to easily and quickly check the results of long and complicated processes of multiplication or division.

The important application of logarithms is in finding the numerical values of numbers that are raised to fractional powers; such for example as $2.8^{1\cdot 6}$; two and eight-tenths raised to the one and six-tenths power. It would be difficult to perform such an operation by ordinary arithmetical processes, but it is a very simple process by use of logarithms.

As used in ordinary practice *the logarithm of any number is the EXPONENT denoting the power to which a number, called the BASE of the system must be raised, to produce the given number.*

Instead of writing the word logarithm, the abbreviation *log* is used.

For example if

$$y = a^x, \text{ and } y' = a^{x'} \text{ then } {}^x \text{ and } {}^{x'}$$

are the logarithms of y and y' in that particular system whose base is **a**; then

$$x = \log\ y, \text{ and } x' = \log\ y'$$

To consider a *numerical* example, let 2 be taken as the *base,* and suppose 64 be a given number.

Then $64=2^6$ and log of $64=6$.

With a base 2, a table of logarithms may be composed as follows:

$2^0= 1$ and $\log 1=0$
$2^1= 2$ and $\log 2=1$
$2^2= 4$ and $\log 4=2$
$2^3= 8$ and $\log 8=3$
$2^4=16$ and $\log 16=4$
$2^5=32$ and $\log 32=5$

The logarithm of 3 in this system is evidently between 1 and 2, and it is 1.584; that is, 2 must be raised to the 1.584 power to equal the number 3.

The raising of 2 to the 1.584 power may be expressed as follows:

$$2^{(1.584)} = 2^{\left(1+\frac{584}{1000}\right)} = 2^{\left(\frac{1584}{1000}\right)} = \sqrt[1000]{2^{1584}}$$

In this system the logarithm of 5 is between 2 and 3; as is also the logarithm of 6, and of 7.

If 4 *be* taken as *a base* then:

$4^0 = 1$, and log $1=0$;
$4^1 = 4$, and log $4=1$; instead of 2 as in the system with base 2.
$4^2 = 16$, and log $16=2$; instead of 4 as in the system with base 2.
$4^3 = 64$, and log $64=3$; instead of 6 as in the system with base 2.

A large number of systems of logarithms might be worked out, using as bases any positive number (whole number or a fraction or a whole number and a fraction) *except* 1.

In any system of logarithms the fractional part of the logarithm of any number is called the *"mantissa"*, (plural mantissae) and the integral part of the logarithm is called the *"characteristic"*.

In the system with 2 as its base, the logarithm of 3 being 1.584, the fraction 0.584 is the *mantissa* of the logarithm of 3; while 1 is the characteristic.

Only *two* systems of logarithms have come into general use,—the one first published by John Spidwell, London, in 1619, called the "natural" system, using as a base **2.718 281 828**; a number obtained by continuing the process indicated by

$$1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{2 \times 3 \times 4 \times 5} \cdots$$

the other, and the one most commonly adopted and used, worked out and published by Henry Briggs in 1624, using 10 as the base, and called the *common* system. From what has been shown regarding the forming of a table of logarithms, it is evident that the logarithm of the same number will be different in the two systems mentioned.

For example: the log of 180 is 2.25527 in the common system, base 10; while the log of 180 in the so-called natural system is 5.1930.

In the common system, **10** is raised to the 2.25527 *power* to amount to 180, while in the natural system **2.718 281**....must be raised to the 5.1930 *power* to equal 180.

To designate which system is employed, the following is adopted: $\log_{10}$ **276**; meaning the logarithm of 276 using the base 10; or $\log_{\epsilon}$ **450**; meaning the logarithm of 450 using the base 2.718281.... The base of the natural system is usually denoted by ϵ; (epsilon).

In general, $\log_a x$ means the logarithm of x, with the base a.

If 10 is used as the base of a system of logarithms the following table of the logarithms of 10 and multiples of 10 could be extended as desired:

Number.	Logarithm.
1	0
10	1
100	2
1000	3
10000	4

The numbers in the second column denotes the power to which 10 must be raised to equal the corresponding number in the first column. For example:

$$10=10^1;\ 100=10^2;\ 1000=10^3.$$

With 10 as a base it is necessary to find the exponent, or the power to which 10 must be raised to give the numbers between 1 and 10, between 10 and 100, between 100 and 1000 and so on.

Such a table has been arranged and printed on page 65. Only the mantissae are printed in this table, as is customary in tables of logarithms employing the base of 10; the characteristic being readily seen by *inspection* of the given number.

The mantissa of the logarithm of 10 as given in the table on page 65, is 000 0000; but since the characteristic is 1, (10 raised to the 1st power=10) the logarithm of 10 is 1.000 0000. The logarithm of 25 is 1.397 9400; the logarithm of 250 is 2.397 9400, and the logarithm of 2500 would be 3.397 9400.

In the table the characteristic of the logarithms of the numbers from 1 to 10 is 0; from 10 to 100 is 1, and from 100 to 1000 is 2.

The logarithm of 24100 could be found by use of the table, by prefixing the proper characteristic to the logarithm of 241.

The log 24100 is 4.3820170. The log 244000 is 5.387 3898.

Another table of logarithms, using the base of 10 is arranged somewhat differently on page 64, showing the common logarithms of a few

numbers from 9950 to 100009. As usual only the fractional part (mantissae) of the logarithm is printed; the integral part (characteristic) being supplied by inspection.

This is a *"seven-place"* table; the first three figures being omitted in all the columns but the second, headed 0, to save space and repetitions.

Also the first two figures of the number are omitted in many cases, in the first column; headed "number". The figures are in different type to facilitate location. The mark $\bar{0}$ indicates that the first three figures of the mantissa *following* the part of the mantissa containing the sign, should be prefixed, instead of the three figures preceding. For example the log 99542 is 4.998 0064; while the log 99540 is 4.9979976. From this table the log 100009 is 5.000 0391.

Number	0		1	2	3	4	5	6	7	8	9
9950	**997**	**8231**	**8274**	**8318**	**8362**	**8405**	**8449**	**8493**	**8536**	**8580**	**8624**
51		8667	8711	8755	8798	8842	8885	8929	8973	9016	9060
52		9104	9147	9191	9235	9278	9322	9365	9409	9453	9496
53		9540	9584	9627	9671	9715	9758	9802	9845	9889	9933
54		9976	$\bar{0}020$	$\bar{0}064$	$\bar{0}107$	$\bar{0}151$	$\bar{0}195$	$\bar{0}238$	$\bar{0}282$	$\bar{0}325$	$\bar{0}369$
55	**998**	*0413*	*0456*	*0500*	*0544*	*0587*	*0631*	*0674*	*0718*	*0762*	*0805*
56		0849	0893	0936	0980	1023	1067	1111	1154	1198	1241
57		1285	1329	1372	1416	1460	1503	1547	1590	1634	1678
58		1721	1765	1808	1852	1896	1939	1893	2026	2070	2114
59		2157	2201	2245	2288	2332	2375	2419	2463	2506	2550
9960	**998**	**2593**	**2637**	**2681**	**2724**	**2768**	**2811**	**2855**	**2899**	**2942**	**2986**
61		3029	3073	3117	3160	3204	3247	3291	3335	3378	3422
62		3465	3509	3553	3596	3640	3683	3727	3771	3814	3858
63		3901	3945	3988	4032	4076	4119	4163	4206	4250	4294
64		4337	4381	4424	4468	4512	4555	4599	4642	4686	4729
65		*4773*	*4817*	*4860*	*4904*	*4947*	*4991*	*5035*	*5078*	*5122*	*5165*
66		5209	5252	5296	5340	5383	5427	5470	5514	5557	5601
67		5645	5688	5732	5775	5819	5862	5906	5950	5993	6037
68		6080	6124	6167	6211	6255	6298	6342	6385	6429	6472
69		6516	6560	6603	6647	6690	6734	6777	6821	6864	6908
9970	**998**	**6952**	**6995**	**7039**	**7082**	**7126**	**7169**	**7213**	**7256**	**7300**	**7344**
71		7387	7431	7474	7518	7561	7605	7648	7692	7736	7779
72		7823	7866	7910	7953	7997	8040	8084	8128	8171	8215
73		8258	8302	8345	8389	8432	8476	8519	8563	8607	8650
74		8694	8737	8781	8824	8868	8911	8955	8998	9042	9086
75		*9129*	*9173*	*9216*	*9260*	*9303*	*9347*	*9390*	*9434*	*9477*	*9521*
76		9564	9608	9651	9695	9739	9782	9826	9869	9913	9956
77	999	0000	0043	0087	0130	0174	0217	0261	0304	0348	0391
78		0435	0479	0522	0566	0609	0653	0696	0740	0783	0827
79		0870	0914	0957	1001	1044	1088	1131	1175	1218	1262
9980	**999**	**1305**	**1349**	**1392**	**1436**	**1479**	**1523**	**1567**	**1610**	**1654**	**1697**
81		1741	1784	1828	1871	1915	1958	2002	2045	2089	2132
82		2176	2219	2263	2306	2350	2393	2437	2480	2524	2567
83		2611	2654	2698	2741	2785	2828	2872	2915	2959	3002
84		3046	3089	3133	3176	3220	3263	3307	3350	3394	3437
85		*3481*	*3524*	*3568*	*3611*	*3655*	*3698*	*3742*	*3785*	*3829*	*3872*
86		3916	3959	4003	4046	4090	4133	4177	4220	4264	4307
87		4350	4394	4437	4481	4524	4568	4611	4655	4698	4742
88		4785	4829	4872	4916	4959	5003	5046	5090	5133	5177
89		5220	5264	5307	5351	5394	5438	5481	5524	5568	5611
9990	**999**	**5655**	**5698**	**5742**	**5785**	**5829**	**5872**	**5916**	**5959**	**6003**	**6046**
91		6090	6133	6177	6220	6263	6307	6350	6394	6437	6481
92		6524	6568	6611	6655	6698	6742	6785	6828	6872	6915
93		6959	7002	7046	7089	7133	7176	7220	7263	7307	7350
94		7393	7437	7480	7524	7567	7611	7654	7698	7741	7785
95		*7828*	*7871*	*7915*	*7958*	*8002*	*8045*	*8089*	*8132*	*8176*	*8210*
96		8262	8306	8349	8393	8436	8480	8523	8567	8610	8653
97		8697	8740	8784	8827	8871	8914	8958	9001	9044	9088
98		9131	9175	9218	9262	9305	9349	9392	9435	9479	9523
99		9566	9609	9653	9696	9739	9783	9826	9870	9913	9957
10000	**000**	**0000**	**0043**	**0087**	**0130**	**0174**	**0217**	**0261**	**0304**	**0347**	**0391**

No.	Logarithm		No.	Logarithm		No.	Logarithm		No.	Logarithm		No.	Logarithm	
0			50	698	9700	100	000	0000	150	176	0913	200	301	0300
1	000	0000	51	707	5702	101	994	3214	151	178	9769	201	303	1961
2	301	0300	52	716	0033	102	008	6002	152	181	8436	202	305	3514
3	477	1213	53	724	2759	103	012	8372	153	184	6914	203	307	4960
4	602	0600	54	732	3938	104	017	0333	154	187	5207	204	309	6302
5	698	9700	55	740	3627	105	021	1893	155	190	3317	205	311	7539
6	778	1513	56	748	1880	106	025	3059	156	193	1246	206	313	8672
7	845	0980	57	755	8749	107	029	3838	157	195	8997	207	315	9703
8	903	0900	58	763	4280	108	033	4238	158	198	6571	208	318	0633
9	954	2425	59	770	8520	109	037	4265	159	201	3971	209	320	1463
10	000	0000	60	778	1513	110	041	3927	160	204	1200	210	322	2193
11	041	3927	61	785	3298	111	045	3230	161	206	8259	211	324	2825
12	079	1812	62	792	3917	112	049	2180	162	209	5150	212	326	3359
13	113	9434	63	799	3405	113	053	0784	163	212	1876	213	328	3796
14	146	1280	64	806	1800	114	056	9049	164	214	8438	214	330	4138
15	176	0913	65	812	9134	115	060	6978	165	217	4839	215	332	4385
16	204	1200	66	819	5439	116	064	4580	166	220	1081	216	334	4538
17	230	4489	67	826	0748	117	068	1859	167	222	7165	217	336	4597
18	255	2725	68	832	5089	118	071	8820	168	225	3093	218	338	4565
19	278	7536	69	838	8491	119	075	5470	169	227	8867	219	340	4441
20	301	0300	70	845	0980	120	079	1812	170	230	4489	220	342	4227
21	322	2193	71	851	2583	121	082	7854	171	232	9961	221	344	3923
22	342	4227	72	857	3325	122	086	3598	172	235	5284	222	346	3530
23	361	7278	73	863	3229	123	089	9051	173	238	0461	223	348	3049
24	380	2112	74	869	2317	124	093	4217	174	240	5492	224	350	2480
25	397	9400	75	875	0613	125	096	9100	175	243	0380	225	352	1825
26	414	9733	76	880	8136	126	100	3705	176	245	5127	226	354	1084
27	431	3638	77	886	4907	127	103	8037	177	247	9733	227	356	0259
28	447	1580	78	892	0946	128	107	2100	178	250	4200	228	357	9343
29	462	3980	79	897	6271	129	110	5897	179	252	8530	229	359	8355
30	477	1213	80	903	0900	130	113	9434	180	255	2725	230	361	7278
31	491	3617	81	908	4850	131	117	2713	181	257	6786	231	363	6120
32	505	1500	82	913	8139	132	120	5739	182	260	0174	232	365	4880
33	518	5139	83	919	0781	133	123	8516	183	262	4511	233	367	3559
34	531	4789	84	924	2793	134	127	1048	184	264	8178	234	369	2159
35	544	0680	85	929	4189	135	130	3338	185	267	1717	235	371	0679
36	556	3025	86	934	4985	136	133	5389	186	269	5129	236	372	9120
37	568	2017	87	939	5193	137	136	7206	187	271	8416	237	374	7483
38	579	7836	88	944	4827	138	139	8791	188	274	1578	238	376	5770
39	591	0646	89	949	3900	139	143	0148	189	276	4618	239	378	3979
40	602	0600	90	954	2425	140	146	1280	190	278	7536	240	380	2112
41	612	7839	91	959	0414	141	149	2191	191	281	0334	241	382	0170
42	623	2493	92	963	7878	142	152	2883	192	283	3012	242	383	8154
43	633	4685	93	968	4829	143	155	3360	193	285	5573	243	385	6063
44	643	4527	94	973	1279	144	158	3625	194	287	8017	244	387	3898
45	653	2125	95	977	7236	145	161	3680	195	290	0346	245	389	1661
46	662	7578	96	982	2712	146	164	3529	196	292	2561	246	390	9351
47	672	0979	97	986	7717	147	167	3173	197	294	4662	247	392	6970
48	681	2412	98	991	2261	148	170	2617	198	296	6652	248	394	4517
49	690	1961	99	995	6352	149	173	1863	199	298	8531	249	396	1993
50	698	9700	100	000	0000	150	176	0913	200	301	0300	250	397	9400

PROBLEM 14: If 3 is given as the base of a system of logarithms find the logarithms of the following numbers: 9; 27; 243; 2187 and 59049.

Answer.

PROBLEM 14a: From the table on page 65 find the logarithm of the following numbers: 9; 27; 82; 180 and 243. Do not fail to supply the proper characteristic in each case.

Answer.

PROBLEM 14b: Using the table on page 65 and prefixing the proper characteristics, write the logarithms of 2700; 181000; 200; 2000, and 2000000.

Answer.

LESSON XV.

GENERAL PROPERTIES OF LOGARITHMS.

Theorem I. *In any system of logarithms the logarithm of unity is zero.*

Let a denote any base, then:

$$a^m \times a^o = a^{(m+o)} = a^m; \text{ from which;}$$

$$a^o = \frac{a^m}{a^m} = 1; \text{ or } a^o = 1$$

$$\therefore \log_a 1 = 0$$

It is often useful in solving equations to know the above fact.

Theorem II. *In any system of logarithms the logarithm of the base itself is unity.*

If a denotes any base, then;

$$a^1 = a; \text{ or } \log_a a = 1$$

If the base is 10 then $10^1 = 10$; or $\log_{10} 10 = 1$ (see page 62).

Theorem III. *In any system of logarithms having a base* **greater** *than unity, the logarithm of zero is minus infinity.*

If base $a > 1$, then $a^{-\infty} + \frac{1}{a^\infty} = \frac{1}{\infty} = 0$ (see page 15 for $>$).

A number raised to a greater and greater power gives a constantly increasing result, provided the given number is greater than 1.

This is seen on page 33 where 10 is raised to different powers.

Any number divided by a number that is too great to be measured must give as a quotient a number too small to be measured.

$$\therefore \log_a 0 = -\infty \text{ (see page 9 for } \infty).$$

Theorem IV. *In any system of logarithms whose base is* **less** *than unity, the logarithm of zero is infinity.*

For if $a < 1$, $a^\infty = 0$, $\therefore \log_a 0 = \infty$.

To show that $a^\infty = 0$ when a is less than 1; suppose $a = \frac{1}{2}$; then $a^2 = \frac{1}{4}$, $a^3 = \frac{1}{8}$, $a^4 = \frac{1}{16}$, $a^5 = \frac{1}{32}$ and the higher the power to which $a = \frac{1}{2}$ is raised, the *less* the numerical value of the result of dividing 1 by the increasing number becomes.

Theorem V. *In* any *system of logarithms, having a positive number for its base, the logarithm of a negative number is imaginary.*

Since the base is positive, no *power* of the *base* can ever become a *negative* number. Many mathematical operations may however be performed with negative numbers, using logarithms, by proceeding as if the numbers were positive and prefixing the proper *sign* at the end of the process.

The *arithmetical complement of* a *logarithm* is the remainder obtained by subtracting the logarithm from **10**.

The practical utility of any system of logarithms is really based upon the two following theorems.

Theorem VI. *In* **any** *system of logarithms, the logarithm of the product of two or more factors is equal to the sum of the logarithms of the factors. If* **a** *denotes the base of the given system then*:

$$\text{let } \left.\begin{aligned} a^x &= m; \\ a^y &= n; \end{aligned}\right\} \text{ from which } \left\{\begin{aligned} &\log_a m = x, \\ \text{and } &\log_a n = y. \end{aligned}\right.$$

then multiplying member by member, $a^x \times a^y = m \times n$, or $a^{(x+y)} = mn$, and $\log_a mn = x+y$. Also adding member to member, $\log_a m + \log_a n = x + y$. And $\mathbf{\log_a mn = \log_a m + \log_a n}$. (Two quantities equal to the same quantity are equal to each other.)

The same theorem may likewise be proved for the product of **three** *or more* factors.

Example 15: Given the $\log_{10}$ of $2 = .30103$ and the $\log_{10}$ of $3 = .47712$, find the $\log_{10}$ of 288.

$$288 = 2\times2\times2\times2\times2\times3\times3$$

But $\log_{10}288 = \log_{10}2 + \log_{10}2 + \log_{10}2 + \log_{10}2 + \log_{10}2 + \log_{10}3 + \log_{10}3$

$$\therefore \log_{10}288 = 5 \times (.30103) + 2 \times (.47712) + 2.45939.$$

Example 15a: Suppose it is desired to *find the logarithm of* **15625.** This number is equal to $25\times25\times25$; therefore the **log 15625** $=\log 25 + \log 25 + \log 25$;

$=1.397\ 9400 + 1.397\ 9400 + 1.397\ 9400$;
$=$**4.193 8200.** (The characteristic 4, is checked by *inspection*).

This example may be checked by the following:

$15625 = 5\times5\times5\times5\times5\times5.$
log $15625 = \log 5 + \log 5 + \log 5 + \log 5 + \log 5 + \log 5.$
$=0.698\ 9700\times6$;
$=4.1938200.$

It may be noted that $15625 = 5^6$ (The sixth power of $5 = 15625$).

This theorem leads to another important principle that may be treated as a separate theorem. This has to do with evaluating the *powers* or the *roots* of any number.

While it is a simple process to find the 6th or any other power of a number, it may require considerable *time;* the use of logarithms in such a case greatly lessens the time and the labor. The same is true as to finding any root, (as the 27th root) of a number.

Theorem VII. To find the logarithm of the power of any number, *multiply the logarithm of the number by the index of the power,* and *the product will be the* **logarithm** *of the power* of the given number.

This is true whether the index is a whole number or a fraction.

Example 15b: Find the **logarithm** of 5^3 from the table on page 65.

$$\log 5 = 0.698\ 9700; \quad 3 \times 0.6989700 = \mathbf{2.0969100.}$$

This may be checked by looking up the log 125 in the same table.

Example 15c: Find the logarithm of $243^{\frac{1}{4}}$. The logarithm of $243 = 2.385\ 6063$; multiplying this logarithm by $\frac{1}{4}$, gives 0.5964016; which is the log $243^{\frac{1}{4}}$.

Theorem VIII. *In* ANY *system of logarithms the logarithm of the quotient of two numbers is equal to the logarithm of the dividend, minus the logarithm of the divisor.*

This theorem may also be expressed as follows:

In ANY *system of logarithms the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Let the base of a given system be denoted by **a**,

$$\left.\begin{array}{r}\text{and let } \mathbf{a}^x = \text{m}, \\ \text{and } \mathbf{a}^y = \text{n},\end{array}\right\} \text{ then } \left\{\begin{array}{l}\log_a \mathbf{m} = x^* \\ \log_a \mathbf{n} = y\end{array}\right.$$

Dividing the above two equations, member by member gives:

$\frac{a^x}{a^y} = \frac{\mathbf{m}}{\mathbf{n}}$ or $a^{(x-y)} = \frac{\mathbf{m}}{\mathbf{n}}$. And $\log_a \frac{\mathbf{m}}{\mathbf{n}} = x - y$. But subtracting the above two equations, member by member, gives $\log_a \mathbf{m} - \log_a \mathbf{n} = x - y$.

$\therefore \log_a \frac{\mathbf{m}}{\mathbf{n}} = \log_a \mathbf{m} - \text{og}_a \mathbf{n}$. (Quantities equal to the same quantity are equal to each other.)

---

* This is true because $x \log \mathbf{a} = \log \mathbf{m}$ and the log, **a** (logarithm of the *base* of the system), is $= 1$. $\therefore x = \log \mathbf{m}$.

Example 15d: Find the logarithm of 248 divided by 62 or the log$\frac{248}{62}$.

log 248=2.395 4517 and log 62=1.792 3917.

log $\frac{248}{62}$=log 248—log 62.

=2.394 4517—1.792 3917.

= .6020600.

The *number* corresponding with this log is found from the table on page 65, to be 4, which is the quotient.

Letting x and y denote any numbers whatever, and **a** the base of the system, the preceding theorems may be recapitulated for handy reference as follows:

$$\log 1=0$$

$$\log \mathbf{a}=1$$

$$\log_{a>1} 0=-\infty$$

$$\log_{a<1} 0=\infty$$

$$\log -x \text{ is imaginary}$$

$$\log xy=\log x+\log y$$

$$\log \frac{x}{y}=\log x-\log y.$$

$$\log x^n=n \log x$$

$$\log \sqrt[n]{x}=\frac{1}{n}\log x$$

$$\log \frac{1}{x}=-\log x$$

Having explained the *theory* of logarithms, and how they may be obtained, a few words will be devoted to the practical application or the use of logarithms. While the logarithm of many numbers could be obtained by applying the principle already mentioned, the use of the table is limited, in finding the logarithm of a number. If a logarithm is given to find the number corresponding to it, the limits of the table are evident. The *larger* the tables the more useful, and this is why they are usually published in a separate volume.

The more figures there are in the logarithms the more accurate the answer obtained by their use. For all ordinary calculations the *"seven-place"* logarithmic table, as illustrated on page 65, is sufficient.

Suppose it is desired to find the numerical value of 20^5 (twenty raised to the fifth power.)

According to Theorem VI, page 68, look up the log 20, multiply it by 5, and from a table of logarithms look up the number corresponding to the logarithm found by multiplying the log 20 by 5. The log 20=1.301 0300 times 5=6.505 1500. The number corresponding with .505 1500 is 32 (see table page 65). Since the characteristic is 5, there must be six digits* in the answer. The answer is therefore 3,200,000. This operation is just the reverse of Theorem VI.

PROBLEM 15: Given the $\log_{10}2=0.30103$; $\log_{10}3=0.4771213$; $\log_{10}4 = 0.60206$, and $\log_{10}7 = 0.845098$, find the logarithm of 32.

Answer.

PROBLEM 15a: Given the logs as in problem 15, find the $\log_{10}$ 196.

Answer.

PROBLEM 15b: Given the logs as in problem 15, find the $\log_{10}$ 336.

Answer.

PROBLEM 15c: Given the logs as in problem 15, find the $\log_{10}$ 886.

Answer.

* See page 61.

PROBLEM 15d: Given the logs as in problem 15, find the $\log_{10}$ 1323.

Answer.

PROBLEM 15e: With the values of logs given in problem 15, find $\log_{10} 3929$.

Answer.

PROBLEM 15f: With the values of logs given in problem 15, find $\log_{10} 9261$.

Answer.

PROBLEM 15g: With the values of logs given in problem 15, find $\log_{10} 37044$.

Answer.

PROBLEM 15h: With the values of logs given in problem 15, find $\log_{10} 54$.

Answer.

PROBLEM 15i: Using the method of logarithms, multiply 8 by 9.

Answer.

PROBLEM 15j: By method of logarithms find the quotient of 225 divided by 15.

Answer.

PROBLEM 15k: Find the numerical value of 22^5, using logarithms.

Answer.

PROBLEM 15l: Find the numerical value of 50^2, using logarithms.

Answer.

PROBLEM 15m: Find the logarithm of the square root of 20.

Answer.

PROBLEM 15n: Find the logarithm of the $\sqrt{15}$.

Answer.

PROBLEM 15o: Find the $\log_{10} 216^{\frac{2}{3}}$.

Answer.

PROBLEM 15p: Find the numerical value of $128^{\frac{1}{7}}$.

Answer.

PROBLEM 15q: Compute the numerical value of the square root of 3.14159. Find the same by logarithms to check result.

Answer.

REVIEW PROBLEMS.

PROBLEM 1: Compute the decimal equivalent of $\frac{3}{128}$.

Answer.

PROBLEM 2: Multiply 654321 by 1234, and prove result by the "American Short Proof Method".

Answer.

PROBLEM 3: Multiply 50^2 by 25^{-2}.

Answer.

PROBLEM 4: Divide 25^2 by 50^2.

Answer.

PROBLEM 5: Express algebraically, the *cube* of **a—c**.

Answer.

PROBLEM 6: Express algebraically, the cube of c—a.

Answer.

PROBLEM 7: Find the numerical value of y, in the equation $y^2 = 36$.

Answer.

PROBLEM 8: Find the numerical value of x, in the equation $x^2 + y^2 = 61$ if $y = 5$.

Answer.

PROBLEM 9: If $R_1 = 5$, and $R_2 = 4$; find the numerical value of R in the equation $R = \frac{R_1 \times R_2}{R_2 + R_1}$

Answer.

PROBLEM 10: If $\rho = 10.79$; $l = 1000$, and $A = 4{,}107$, find the value of R in the equation $R = \rho \frac{l}{A}$.

Answer.

PART II.

The following *examples* and explanations have been assembled and added to this book for the purpose of assisting high school pupils who desire an opportunity to receive help outside of school instruction, and also to aid any who have had limited opportunity for school instruction but who desire to pursue the study of Algebra by themselves. The following examples and explanations will be of considerable aid to those who propose to take college entrance examinations. Several college entrance examinations have been added to the text, and the problems properly solved.

While many of the *problems* presented for solution in ordinary text books dealing with Algebra are absolutely without practical application, one should not ignore the importance of the mental training one receives from their logical and careful solution.

One *unconsciously* employs mental training at many times in one's career, and many years of careful and continuous practice are needed to realize the attainment of a notable success that is effected in a few moments. In other words, while an individual's reputation is apparently made in a few short moments, it has required years of patient and continuous labor to enable the individual to successfully meet brief demands at a critical time.

Students who study the examples and problems presented throughout this book should constantly keep in mind that these are *typical,* serving as a guide in the process of solving similar problems, and that many problems presenting different *numerical* values, may be readily solved by a proper substitution of values, in the type forms here presented.

LESSON XVI

DEFINITIONS.

Examples of different kinds of equations were cited on page 53 and several rules given pertaining to the solution of equations. There are a variety of algebraical equations, and a few of these will be given particular consideration in the following portion of the text.

Symmetrical Equations, are those which are not affected by any interchange of the unknown quantities.

$x^2 + 2xy + y^2 = 25$ is a symmetrical equation.
$a - b = 4$ is not a symmetrical equation.

Homogeneous Equations, are equations having their *terms* all of the same degree, as regards the unknown quantities.

$x^2 + 2xy + y^2 = 25$, $x^2 - 4xy - 16y^2 = 10$, and
$x^2 + y^2 = xy$ are homogenous equations.

Literal Equations, are equations whose terms consist entirely of *letters*. $x^2 + y^2 = xy$, $ax + bx = mn$, and
$\frac{a}{x} - \frac{b}{x} = c$ are all examples of literal equations.

Absolute Term: The so-called *absolute term* of any equation is the term which does not contain any unknown quantity.

In the equation $x^2 + 2xy + y^2 = 25$, the absolute term is 25.

Every equation may be considered to be the expression, in algebraic language, of a particular question.

Thus the equation $x + x = 24$ is the algebraic expression of the following question:

What number added to itself will give 24?

If the answer to this question be required it might be arrived at by the following process:

By adding x to itself

$2x = 24$, and dividing both members of the equation by 2;
$x = 12$, results.

The solution of a question by algebraic methods consists of two distinct parts:

1st. *To make the* STATEMENT: *that is to express the condition of the question algebraically.*

2nd. *To solve the equation: or in other words to find a numerical value from the unknown quantities; or to separate the known from the unknown quantities.*

Unfortunately no specific rules can be given to guide in the algebraical statement of questions or problems. It is necessary to carefully study the given problem to see if the algebraical equation is not immediately evident, or if it is not possible to discover new conditions from which an equation may be formed.

The best method to pursue in order to become expert in stating problems, is to state as many as possible.

The solution of problems, after the statement has been made, is comparatively easy.

LINEAR EQUATIONS: EQUATIONS OF THE FIRST DEGREE

EXAMPLE 16: A cistern is filled by a pipe, and the amount let into the cistern during 8 minutes is 45 gallons more than the quantity let in during five minutes. Find the number of gallons let in per minute.

Let x denote the number of gallons let in during 5 minutes.

Then $x + 45$ will denote the number of gallons let in during 8 minutes.

The average input, or the number of gallons let in per minute will be expressed by;

$\frac{x+45}{8} = \frac{x}{5}$; from which is obtained,

$$5x + 225 = 8x$$
$$x = 75$$

The number of gallons let into the tank per minute, is therefore 15. In 8 minutes there would be $8 \times 15 = 120$ gallons let in, and in 5 minutes there would be $5 \times 15 = 75$ gallons. $120 - 75 = 45$.

PROBLEM 16: A cistern being filled by a pipe, the amount let into the cistern during the first 8 minutes is 60 gallons more than the quantity let in during the next 4 minutes. Find the number of gallons let in per minute. **15 gallons. Answer.**

Problem 16a: While a cistern is being filled, the amount let in during the first 10 minutes is 50 gallons more than the quantity let in during the next 4 minutes. Find the number of gallons let in per minute.

EXAMPLE 17: A man is 33 years old, and his son is 12 years old. How many years ago was the father four times as old as the son?

Let x denote the number of years ago when the father was 4 times as old as his son.

Then; $33 - x = 4\,(12 - x)$

and $33 - x = 48 - 4x$

$$3x = 15$$
$$x = 5$$

That is, 5 years ago the father was four times as old as his son.

Proof:

$$33 - 5 = 28.$$
$$12 - 5 = 7.$$
$$4 \times 7 = 28.$$

PROBLEM 17: A man is 60 years old, and his son is 20 years old. How many years ago was the father five times as old as the son;

10 years ago. Answer.

Problem 17a: A man is 50 years old, and his son is 15 years old. How mnny years ago was the father six times as old as the son?

EXAMPLE 18: A man is 36 years old, and his son is 11 years old. In how many years will the father be twice as old as his son.

Let x denote the *number of years hence,* when the father will be twice as old as his son.

Then; $36 + x = 2\,(11 + x)$

$$36 + x = 22 + 2x$$
$$x = 14$$

In 14 years the father will be twice as old as his son.

Proof:

$$36 + 14 = 50$$
$$11 + 14 = 25$$
$$50 = 2 \times 25$$

PROBLEM 18: A man is 50 years old and his son is 20 years old. In how many years will the father be twice as old as the son?

In 10 years. Answer.

Problem 18a: A man is 49 years old and his son is 18 years old. In how many years will the father be twice as old as the son?

EXAMPLE 19: A has \$20, and B has \$30. How many dollars must B give to A in order that A shall have four times as much as B has?

Let x denote the number of dollars that B must give to A.

Then;

$$20 + x = 4\,(30 - x)$$
$$20 + x = 120 - 4x$$
$$5x = 100$$
$$x = 20$$

Or B must give \$20 to A.

Proof:

$$20 + 20 = 40$$
$$30 - 20 = 10$$
$$40 = 4 \times 10$$

PROBLEM 19: A *has \$100, and* B *has \$1000. How many dollars must* B *give to* A *in order that* A *shall have 5 times as much money as* B *has?*

\$816 $\frac{2}{3}$. Answer.

Problem 19a: A has \$25, and B has \$35. How many dollars must B give to A in order that A shall have three times as many dollars as B has?

EXAMPLE 20: Find two consecutive numbers whose sum is 243.

Let x denote one number, and $x + 1$ denote the other.

Then;

$$x + (x + 1) = 243$$
$$2x = 242$$
$$x = 121$$

One number is 121, the other is $121 + 1 = 122$.

Proof: $121 + 122 = 243.$

PROBLEM 20: Find the numerical value of two consecutive numbers whose sum is 267. **133 and 134. Answer.**

Problem 20a: Find the numerical value of two consecutive numbers whose sum is 365.

Example 21: The difference of two numbers is 8, and their sum is five times the smaller. Find the two numbers.

Let x denote the smaller number.

Then $x+8$ denotes the larger number, and according to the given conditions;

$$x+(x+8)=5x$$
$$2x+8=5x$$
$$3x=8$$
$$x=\tfrac{8}{3}=2\tfrac{2}{3}.$$

The smaller number is therefore $2\frac{2}{3}$ and the larger number is $2\frac{2}{3}+8=10\frac{2}{3}$.

PROBLEM 21: The difference of two numbers being 20, and their sum being 4 times the smaller, find the two numbers.

Smaller = 10, and larger = 30. Answer.

Problem 21a: The difference of two numbers being 35 and their sum being 9 times the smaller, find the value of the two numbers.

EXAMPLE 22: The night at Petrograd (St. Petersburg) on December 21, lasts 13 hours longer than the day. What is the duration of the day, in hours?

Let x denote the length, in hours, of the day.

Then $24-x$ denotes the length of the night, in hours.

By condition;

$$24-x=x+13$$
$$2x=11$$
$$x=\tfrac{11}{2}=5\tfrac{1}{2}\text{ hours.}$$

The day is therefore only $5\frac{1}{2}$ hours in length.

PROBLEM 22: The night at Chicago on December 21, lasts $4\frac{1}{2}$ hours longer than the day. What is the duration of the day, in hours?

$9\frac{3}{4}$ hours. Answer.

Problem 22a: The night at New Orleans, on December 21, lasts 5 hours longer than the day. What is the duration of the day, on the same date, in hours?

EXAMPLE 23: A line that is 45 inches long is divided into two parts. Two times the longer part exceeds three times the shorter part by 30 inches. How many inches are there in each part?

Let x denote the longer part.

Then $45 - x$ denotes the shorter part.

By condition; $$2x - 30 = 3(45 - x)$$
$$2x - 30 = 135 - 3x$$
$$5x = 165$$
$$x = 33$$

The longer part is 33 inches long.

The shorter part is therefore $45 - 33 = 12$ inches long.

Proof: $$2 \times 33 - 30 = 3 \times 12$$
$$66 - 30 = 36$$

PROBLEM 23: How may a line 77 inches long be divided into two parts one of which is $2\frac{1}{2}$ times as long as the other?

Shorter is 22 inches long, and longer is 55 inches long.

Answer.

Problem 23a: How may a line 55 inches long be divided in two parts such that two times the longer part exceeds three times the shorter part by 20?

EXAMPLE 24: The difference of the squares of two numbers is 221; if their sum is 17, what are the numbers?

Let x denote the larger number.

Then $17 - x$ will denote the smaller number.

By the conditions imposed in the example;

$$x^2 - (17 - x)^2 = 221$$ (See page 46).

Then; $x^2 - (289 - 34x + x^2) = 221$, after changing signs of all terms in parentheses,

$$x^2 - x^2 + 34x - 289 = 221$$

From which; $$34x = 221 + 289 = 510$$

$$x = \frac{510}{34} = 15$$

The larger number being 15, the smaller is $17 - 15 = 2$.

Proof: $15^2 - 2^2 = 225 - 4 = 221$.

PROBLEM 24: The difference of the squares of two numbers being 400, and the sum of the two numbers being 40, find each of the two numbers. **One is 15; the other 25. Answer.**

Problem 24a: The difference in the squares of two numbers is 7 and the sum of the two numbers is 7; find the value of each of the numbers.

EXAMPLE 25: A has \$16 less than B. If B gives \$20 to A, then A will have five times as many dollars as B. How many dollars has each?

Let x denote the number of dollars that A has.
Then $x + 16$ will denote the number of dollars B has.
And by the given conditions; $x + 20 = 5(x + 16 - 20) = 5(x - 4)$

$$x + 20 = 5x - 20$$
$$4x = 40$$
$$x = 10$$

Therefore A has \$10, and B has $10 + 16 = \$26$.
Proof: $26 - 20 = 6$; $10 + 20 = 30$. $30 = 6 \times 5$.

EXAMPLE 26: A began in business with three times as much capital as B. During the first year A lost \$600, B gained \$200, and A had then only twice as much as B. What amount of capital did each start with?

Let $3x$ denote A's capital; in dollars
Then x will denote B's capital at the start; in dollars.

By condition; $3x - 600 = 2(x + 200)$
$$3x - 600 = 2x + 400$$
$x = \$1000$. B started with \$1000 and A started with \$3000.

EXAMPLE 27: A man has \$6.25 in half dollars and quarters. He has three times as many quarters as he has half dollars. How many half dollars, and how many quarters has he?

Let x denote the number of half dollars.
Then $3x$ denotes the number of quarters.

Then by condition: $50x + 25(3x) = 625$
(dividing by 25) $2x + 3x = 25$
$$5x = 25$$
$$x = 5$$

That is the man has 5 half dollars and $3\times5=15$ quarters.

Proof:

$$\begin{array}{r} 50\times\ 5=250 \\ 25\times15=375 \\ \hline 625 \end{array} \text{ cents or \$6.25}$$

EXAMPLE 28: The amount of $3.50 is made up of half dollars and dimes. If there are 19 coins altogether, how many are there of each coin?

Let x denote the number of half dollars.
Then will $19-x$ denote the number of dimes.

By condition; $\quad 50x+10\,(19-x)=350$ cents

(dividing by 10) $\quad 5x+19-x=35$

$$4x=16$$
$$x=4$$

There are 4 half dollars; therefore there are $19-4=15$ dimes.

Proof:

$$\begin{array}{r} 4\times50=200 \\ 15\times10=150 \\ \hline 350 \end{array} \text{ cents or \$3.50}$$

LESSON XVII

EXAMPLE 29: A merchant adds to his capital one-fourth of it each year. At the end of each year he deducts \$1200 for expenses. At the end of the third year he has, after the deduction of the last \$1200, one and a half times his original capital, minus \$950. What was his original capital?

Let x denote his original capital.

Then will $x+\frac{1}{4}x$ denote the amount at the end of the first year, and $(x+\frac{1}{4}x)+\frac{1}{4}x$ denote the amount at the end of the second year.

By the condition stated in the example;

$$x+\tfrac{3}{4}x-3\ (1200)=1\tfrac{1}{2}x-950$$

From which is obtained;

$$\begin{aligned} x+\tfrac{3}{4}x-\tfrac{3}{2}x &= 3600-950 \\ \tfrac{4}{4}x+\tfrac{3}{4}x-\tfrac{6}{4}x &= 2650 \\ \tfrac{7}{4}x-\tfrac{6}{4}x &= 2650 \\ \tfrac{1}{4}x &= 2650 \\ x &= 10600 \end{aligned}$$

His original capital was theretore \$10,600.

To verify this result proceed as follows;

$3 \times \$1200 = \$3600$ expenses for three years.

$\frac{\$10600}{4} \times 3 = \$7,950$ total additions to capital.

$\$7,950 - \$3,600 = \$4,350$ *net* addition to capital.

$\$10,600 + \$4,350 = \$14,950$

$\$14,950 + \$950 = \$15,900$ which is $1\frac{1}{2}$ of original capital.

EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

Whenever there are two unknown quantities in a given equation, one of them may be *eliminated* by substituting its value in terms of the other unknown quantity.

EXAMPLE 30: Given a room of such dimensions, that the difference of the sides multiplied by the lesser side is equal to 36, and the product of the sides is equal to 360. Find the numerical value of each side.

Let x denote the lesser side.
And y denote the greater side.
Then by the first condition; $(y-x)\,x=36$,
and by the second condition; $xy=360$
From the first equation is obtained;

$$xy-x^2=36$$

Solving the second equation for y gives; $y=\dfrac{360}{x}$

Substituting this value of y, in the first equation gives;

$$x\frac{360}{x}-x^2=36.\quad \text{From which is obtained; } x^2=324$$

The value of x is therefore $\pm\sqrt{324}=\pm 18$.

If $x=\pm 18$, then by substituting this value in $xy=360$ is obtained; $\pm 18y=360$, $y=\dfrac{360}{18}=\pm 20$.

The same result could also have been obtained by subtracting $xy-x^2=36$ from $xy=360$ as follows;

$$\begin{array}{r@{\,=\,}r} xy & 360 \\ +x^2-xy & -\ 36 \\ \hline x^2 & 324 \\ x & \pm\ 18 \end{array}$$

The shorter side is 18, and the longer is 20 feet long.

EXAMPLE 31: What two numbers have a product equal to 30 and quotient equal to $3\frac{1}{3}$?

Let x denote one number.
And y denote the other number.
Then by the first condition;

$$xy=30$$

and by the second condition, $\frac{x}{y}=3\frac{1}{3}=\frac{10}{3}$

From the first equation $x=\dfrac{30}{y}$ and from the second equation $x=\frac{10}{3}y$.

It is therefore evident that

$$\tfrac{10}{3}y=\frac{30}{y}$$

$$y^2=\tfrac{90}{10}=9$$

Hence; $y=3$.

Substituting in the first equation the value 3 just obtained for y gives

$$x3=30$$

$$x=\tfrac{30}{3}=10.$$

Verification of the foregoing;

$$3\times 10=30, \text{ and } \tfrac{10}{3}=3\tfrac{1}{3}.$$

It seems to be common experience that equations and problems involving *literal* factors and terms are most difficult of solution, therefore a number of typical cases are presented.

EXAMPLE 32: If the product of two numbers is *a* and their quotient *b*, find the numbers.

Let x denote one number.

And y denote the other number.

Then by the first condition,

$xy=a$, and by the second condition,

$$\frac{x}{y}=b.$$

From the first equation $x=\frac{a}{y}$ and from the second equation $x=by$.

Therefore, $$by=\frac{a}{y}$$

$$y^2=\frac{a}{b}$$

Hence, $$y=\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

And by properly substituting in the first equation the value of y just obtained;

$$x\frac{\sqrt{a}}{\sqrt{b}}=a$$

$$x=\sqrt{b}\frac{a}{\sqrt{a}}=\sqrt{a}\sqrt{b}$$

EXAMPLE 33: If the sum of the squares of two numbers is *a*, and the difference of their squares is *b*, what are the numbers?

Let x denote one number.

And y denote the other number.

Then by the first condition;

$x^2+y^2=a$, and by the second condition;

$x^2-y^2=b$.

By adding the two equations;

$$2x^2=a+b$$

$$x^2=\frac{a+b}{2}$$

$$x=\sqrt{\frac{a+b}{2}}$$

By subtracting the two equations;

$$2y^2=a-b$$

$$y^2=\sqrt{\frac{a-b}{2}}$$

EXAMPLE 34: Find two numbers which are to each other as *m* to *n*, and the sum of whose squares is equal to a^2.

Let x denote one number.

And y denote the other number.

Then $\frac{x}{y}=\frac{m}{n}$ and $x^2+y^2=a^2$

From the first equation; $x=\frac{m}{n}y$ and from the second equation $x=\pm\sqrt{a^2-y^2}$.

Therefore, $\frac{m}{n}y=\pm\sqrt{a^2-y^2}$.

Squaring both members gives;

$$\frac{m^2}{n^2}y^2=a^2-y^2$$

And solving for y gives;

$$\frac{m^2}{n^2}y^2+y^2=a^2$$

$$y^2\left(1+\frac{m^2}{n^2}\right)=a^2$$

$$y^2\ \frac{n^2+m^2}{n^2}=a^2$$

$$y^2=\frac{a^2n^2}{n^2+m^2}$$

$$y=\pm\sqrt{\frac{a^2n^2}{n^2+m^2}}=\pm\frac{a\,n}{\sqrt{m^2+n^2}}$$

Substituting in the first equation the value obtained for y gives;

$$\frac{x}{\frac{a\,n}{\sqrt{m^2+n^2}}} = \frac{m}{n}$$

From which is obtained; $$x = \frac{a\,n}{\sqrt{m^2+n^2}} \times \frac{m}{n}$$

$$= \frac{a\,m}{\sqrt{m^2+n^2}}$$

EXAMPLE 35: What are the two numbers that are to each other as *m* to *n*, and the difference of whose squares is equal to b^2?

Let x denote one number.

And y denote the other number;

Then $\frac{x}{y} = \frac{m}{n}$ and $x^2 - y^2 = b^2$.

$$x = \frac{m}{n} y \text{ or } x^2 = \frac{m^2}{n^2} y^2$$

Substituting the value for x^2, just found, in the second equation will give;

$$\frac{m^2}{n^2} y^2 - y^2 = b^2$$

$$y^2 \left(\frac{m^2 - n^2}{n^2}\right) = b^2$$

$$y^2 = \frac{b^2\,n^2}{m^2 - n^2}$$

$$y = \pm\sqrt{\frac{b^2\,n^2}{m^2 - n^2}} = \frac{b\,n}{\pm\sqrt{m^2 - n^2}}$$

By proper substitution; $$x = \frac{m\,b}{\pm\sqrt{m^2 - n^2}}$$

If numerical values are assigned to b, m, and n, it may be observed that m^2 must be greater than n^2, in order that their difference shall be a *positive* number. Otherwise the $\sqrt{m^2 - n^2}$ will mean the square root of a negative number.

EXAMPLE 36: A person distributes a sum of money among a number of boys and girls. The number of boys is to the number of girls as 3 to 4. The girls receive one-half as many dollars as there are individuals, and the boys receive twice as many dollars as there are girls. Altogether they receive \$138. How many boys and how many girls were there?

Let x denote the number of boys.

And y denote the number of girls.

Then by condition; $\dfrac{x}{y}=\dfrac{3}{4}$

$$\frac{x+y}{2}+2y=138$$

From the first equation; $x=\frac{3}{4}y$

The second equation may be reduced to;

$$x+y+4y=276$$
$$x+5y=276$$

Substituting in the last equation the value $x=\frac{3}{4}y$, gives;

$$\tfrac{3}{4}y+5y=276$$
$$5\tfrac{3}{4}y=276$$
$$\tfrac{23}{4}y=276$$
$$y=\frac{1104}{23}=48$$

Hence there are 48 girls. Since there are $\frac{3}{4}$ as many boys as girls, there are $\frac{3}{4}\times48=36$ boys.

LINEAR EQUATIONS WITH THREE UNKNOWN QUANTITIES.

EXAMPLE 37: Find three numbers such that the second is three times the first, the third being four times the first, while the difference between the second and the third is five.

Let x denote the first number.
Then $3x$ denotes the second number.
And $4x$ denotes the third number.

By condition; $$4x-3x=5$$ $$x=5$$

The first number being 5, the second is 15, and the third is $4\times5=20$.
Proof: $20-15=5$.

EXAMPLE 38: Find three numbers such that the sum of the first two is 14, the third being twice the first, while the third is greater than the second by 4.

Let x denote the first number.
Then $14-x$ denotes the second number.
And $2x$ denotes the third number.

By condition; $$2x-4=14-x$$ $$3x=18$$ $$x=6$$

The first number is 6; the second is $14-6=8$, and the third is $2\times6=12$.

PROBLEM 38: Find three numbers such that the second number exceeds the first number by 3, the third exceeds the first by 10, while the third is twice the first.

First = 10; second = 13; third = 20. Answer.

PROBLEM 39: Find each of three numbers, such that the sum of the first and second is 5, the sum of the first and third is 6, while the third is twice the first.

First = 2, second = 3, third = 4. Answer.

PROBLEM 40: If the difference between two numbers is 4, while a third number is 5 less than the sum of the first and second, and the sum of the first and third numbers is 14, what are the numbers?

First number is 5, second is 9, and third is 9. Answer.

EXAMPLE 41: A man has 5 sons each of which is three years older than the next younger. If the age of the oldest three years hence will be three times the present age of the youngest, what is the age of each son?

Let x denote the present age of the oldest son.

Then; $x-3$ denotes the age of the second.
$x-6$ " " " " " third.
$x-9$ " " " " " fourth.
$x-12$ " " " " " fifth.

By conditions;
$$x+3=3\ (x-12)$$
$$x+3=3x-36$$
$$2x=39$$
$$x=19\tfrac{1}{2}$$

The oldest son is $19\frac{1}{2}$ years old.
The second son is $16\frac{1}{2}$ " "
The third son is $13\frac{1}{2}$ " "
The fourth son is $10\frac{1}{2}$ " "
The fifth son is $7\frac{1}{2}$ " "

Proof: $19\frac{1}{2}+3=3\times 7\frac{1}{2}$.

PROBLEM 41: A man has four sons each of which is four years older than the next younger. If the age of the oldest three years hence will be three times the present age of the youngest, what is the present age of each?

Oldest is $19\frac{1}{2}$; next is $15\frac{1}{2}$; next is $11\frac{1}{2}$, and youngest is $7\frac{1}{2}$. Answer.

Problem 41a: A man has three sons each of which is four years older than the next younger. If the age of the youngest three years hence is twice the present age of the oldest, what is the present age of each son;

EXAMPLE 42: The sum of the three angles of any triangle is 180 degrees (180°). If the second angle of a triangle is 4 degrees (4°) larger than the first, and the third is twice the sum of the first and second, find the value, in degrees, of each angle.

Let x denote the first angle.
Then $x+4$ will denote the second angle.
And $2\,[x+(x+4)\,]$ will denote the third angle.
By the stated condition; $x+(x+4)+2\,[x+(x+4)\,]=180$

$$2x+4+4x+8=180$$
$$6x=168$$
$$x=28$$

The first angle is 28°; the second angle is therefore $28 + 4 = 32°$, and the third angle is $2(28 + 32) = 2 \times 60 = 120$. The sum of the three angles should be 180°.

$$28° + 32° + 120° = 180°$$

PROBLEM 42: If the second angle of a triangle is 5° larger than the first, and the third angle is three times the sum of the first and second, find the value of each angle of the triangle.

$$x + (x + 5) + 3[x + (x + 5)] = 180$$

First angle = 20°; second angle = 25°; third angle = 135°. Answer.

Problem 42a: If one angle of a triangle is 6° larger than another, while a third angle is 4 times the sum of the other two angles, find the value of each angle of the triangle.

EXAMPLE 43: Find the value of each of three consecutive numbers whose sum is 216.

Let x denote one number.
Then $(x + 1)$ will denote the next consecutive number.

And $(x + 2)$ will denote the next number.
By condition; $x + (x + 1) + (x + 2) = 216$

$$3x = 213$$
$$x = 71$$

One number is 71; the next is 72, and the next is 73.

Proof: $71 + 72 + 73 = 216$.

PROBLEM 43: Find the value of four consecutive numbers whose sum is 26. **5; 6; 7, and 8. Answer.**

Problem 43a: Find the value of each of four consecutive numbers whose sum is 10.

EXAMPLE 44: The sum of three numbers is 100. If the first and second are respectively 24 and 11 greater than the third, what are the numbers?

Let x denote the value of the third number.
Then will $x+11$ denote the value of the second.
And $x+24$ will denote the value of the third.

By condition; $\quad x+x+11+x+24=100$

$$3x=100-35=65$$

$$x=21\tfrac{2}{3}$$

The numbers are therefore $45\frac{2}{3}$, $32\frac{2}{3}$, and $21\frac{2}{3}$. Answer.

PROBLEM 44: The sum of three numbers is 100. If the first and second are 40 and 30 respectively greater than the third, find the numerical value of the third number.

Third number $=10$. Answer.

SOLUTION OF EQUATIONS BY FACTORING

EXAMPLE 45: Solve by factoring $12x^3-7x^2-10x=0$.

The factors are; $\quad (4x^2-5x)\ (3x+2)=0$

$$3x+2=0$$

$$x=-\tfrac{2}{3}$$

Also; $\quad 4x^2-5x=0$

$$x^2-\tfrac{5}{4}x=0$$

$x^2=\frac{5}{4}x$. Dividing both members of this equation by x gives; $x=\frac{5}{4}$.

Another arrangement might be as follows;

$x\,(12x^2-7x-10)=0 \quad$ from which $x=0$.

Also; $\quad 12x^2-7x=10$

$$x^2-\tfrac{7}{12}x=\tfrac{10}{12}$$

Completing the square; $\quad x^2-\frac{7}{12}x+(\frac{7}{24})^2=\frac{10}{12}+(\frac{7}{24})^2=\frac{529}{576}$

$$x-\tfrac{7}{24}=\pm\sqrt{\tfrac{529}{576}}=\pm\tfrac{23}{24}$$

$$x=\tfrac{7}{24}\pm\tfrac{23}{24}=\tfrac{30}{24} \text{ or } -\tfrac{16}{24}$$

$$=\tfrac{5}{4} \text{ or } -\tfrac{2}{3}$$

The three correct values of x are therefore $-\frac{2}{3}$, $\frac{5}{4}$ and 0. Each of these values substituted in the original equation will satisfy the equation.

It is evident that the method of solving equations by factoring, involves much less time and mental effort than the method of "completing the square".

SIMULTANEOUS EQUATIONS

EXAMPLE 46: Solve for *x* and *y* the simultaneous equations;

$$x - \frac{2y - x}{23 - x} = 20 + \frac{2x - 59}{2}$$

$$y - \frac{y - 3}{x - 18} = 30 - \frac{73 - 3y}{3}$$

The first equation may be reduced to;

$$\frac{23x - x^2 - 2y + x}{23 - x} = \frac{40 + 2x - 59}{2} = \frac{2x - 19}{2}$$

$$- 2x^2 - 4y + 48x = 46x + 19x - 2x^2 - 437$$

$$- 4y + 48x = 46x + 19x - 437$$

$$- 17x = 4y - 437$$

$$x = \frac{- 4y + 437}{17}$$

The second equation may be reduced to;

$$\frac{xy - 18y - y + 3}{x - 18} = \frac{90 - 73 + 3y}{3} = \frac{17 + 3y}{3}$$

$$3xy - 54y - 3y + 9 = 17x - 306 + 3xy - 54y$$

From which; $17x = 9$

$$x = \tfrac{9}{17}$$

LESSON XVIII

EXTRACTION OF THE SQUARE ROOT OF NUMBERS

The mention of square roots and squares of numbers was made on pages 46 and 47, where the *square* or second power of a number was stated to be the product resulting from multiplying any number by itself *once* and the *square root* of a number to be that number which multiplied by itself once, will produce the given number.

Evolution and Involution: The process of finding the square root of any number is called *evolution,* while the reverse process: finding the square of any number, is called *involution.*

Although there is no number of which any given power may not be found *exactly,* there are many numbers of which exact roots can not be obtained; but, by the use of decimals, may be *approximately* evaluated to any assigned degree of exactness.

Rational Roots and Surds: Approximate roots are called *surd roots;* while perfect, or accurate roots are called *rational roots.*

While the square of any number whether a whole number or a fraction is always easily found by multiplying the number (either whole or fractional) by itself once, the extraction or obtaining the square root of a number is sometimes attended with difficulty and requires particular explanation.

The first ten numbers are:

1 2 3 4 5 6 7 8 9 10

and their squares are:

1 4 9 16 25 36 49 64 81 100

From which it may be seen that the numbers in the second line are the squares of corresponding numbers in the first line; while the numbers in the first line are exact square roots of corresponding numbers in the second line.

It may be noted that the square root of a number whose value is between two numbers in the second line, will be between two corresponding square roots appearing in the first line.

For example: the square root of 42, that occurs between 36 and 49, will be between 6 and 7. Likewise the square root of 22, which occurs between 16 and 25, will be a number between 4 and 5.

The square of a number consisting of a single figure, will consist of two figures; or the square of a number consisting of but one figure, will consist of a number containing no figure of a higher denomination than tens. Also the square of a number expressed by two figures, will be a number containing no higher denomination than thousands.

The largest number expressed by a single figure is 9, and the square of 9 is 81. The largest number expressed by two figures is 99, and the square of 99 is 9801.

The square of 999 is 998,001.
The square of 9999 is 99,980,001.

For every figure in the given number there are two figures in its square.

Every number of two figures, may be considered as made up of a certain number of *tens* and a certain number of *units*. Thus 81 is made up of 8 tens and 1 unit, and may be expressed in the form of $80+1$.

If now the tens be represented by a and the units by b, the following will be true:

$$a+b= 81$$
$$(a+b)^2=(81)^2$$

or $a^2+2ab+b^2=6561$; which proves the square of a number composed of tens and units, contain *the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

The preceding example may be expressed as follows:

If $a=80$ and $b=1$

$$\begin{aligned} a^2 &= 6400 \\ 2ab &= 160 \\ b^2 &= 1 \\ \hline & \,6561 \end{aligned}$$

Suppose a cipher is added to each of the figures in the first line on page 97 giving:

10 20 30 40 50 60 70 80 90 100

and the squares are:

100 400 900 1600 2500 3600 4900 6400 8100 10000

From which it is seen that the square of one ten is 100, the square of two tens is 400; and in general, that *the square of tens will contain no figure of a less denomination than hundreds, nor of a higher denomination than thousands.*

To find the square root of 6724:

$$
\begin{array}{r r}
 & 8\quad 2 \\
 & \overset{\frown}{67}\ \overset{\frown}{24} \\
 & 64\ 00 \\
\cline{2-2}
160 & 324 \\
2 & \\
\cline{1-1}
162 & 324 \\
\cline{2-2}
\end{array}
$$

It is to be noted that 324 consists of twice the product of the tens by the units plus the square of the units.

The operation that has in reality been performed is subtracting from the number 6724 the square of 8 tens or 80; ($80^2 = 6400$) twice the product of the tens by the units; ($2 \times 80 \times 2 = 320$) and finally the square of the units; ($2^2 = 4$) that is the three components that enter into the composition of the square of $80 + 2$, and since the result of the subtraction is 0, it follows that the square root of 6724, is 82.

To find the square root of 675684:

$$
\begin{array}{r r}
 & 8\quad 2\quad 2 \\
 & \overset{\frown}{67}\ \overset{\frown}{56}\ \overset{\frown}{84} \\
 & 64\ 00\ 00 \\
\cline{2-2}
160 + 2 = \underline{162} & 356\ 00 \\
 & 324\ 00 \\
\cline{2-2}
1640 + 2 = \underline{1642} & 32\ 84 \\
 & 32\ 84 \\
\cline{2-2}
\end{array}
$$

For the extraction of the square root of any number, the following general rule applies:

RULE.

I. *Separate the number into periods of two figures each, starting at the right hand* (the extreme period on the left may often contain but *one* figure).

II. *Find the greatest square contained in the first period on the left and place its root figure over the period, similar to the figure of a quotient in division. Subtract the square of the root from the first period, and to the remainder add the second period for a dividend.*

III. *Double the root number already found, and place it on the left for a divisor. Ascertain how many times the divisor is contained in the dividend, excluding the right hand figure, and place the figure, denoting the number of times, with the root number already found, and also add it to the trial divisor.*

IV. *Multiply the divisor thus constituted, by the last figure of the root number, and subtract the product from the dividend, and to the remainder add the figures of the next period for a new dividend. In case any of the products should be greater than the dividend, diminish the last figure of the root number.*

V. *Double the whole root number already found, for a new divisor, and continue the operation as before, until all the periods have been employed.*

It should be noted as a general principle, that the number of figures in the square root number, will always be equal to the number of periods into which the given number is separated.

As the student will receive greater benefit from the careful study of examples showing the principle pertaining to the extraction of the square root of numbers, than by memorizing the *"rules"* given, a number of typical examples and problems will be given.

EXAMPLE 47: Find the square root of 36729.

$$
\begin{array}{r|l}
 & 1\ \ 9\ \ 1\,.\,6\ \ 4 \\
 & 3\ \overset{\frown}{67}\ \overset{\frown}{29}.\overset{\frown}{00}\ \overset{\frown}{00} \\
 & 1 \\ \cline{2-2}
\underline{2\ \ 9} & 267 \\
 & 261 \\ \cline{2-2}
\underline{38\ 1} & \ \ 629 \\
 & \ \ 381 \\ \cline{2-2}
\underline{382\ 6} & \ \ 24800 \\
 & \ \ 22956 \\ \cline{2-2}
\underline{3832\ 4} & \ \ \ 184400 \\
 & \ \ \ 153296 \\ \cline{2-2}
 & \ \ \ \ 31104 \text{ Remainder.}
\end{array}
$$

In this example there are two periods of decimals, which gives two figures in the decimal portion of the square root number.

As there is a remainder, the root number is only approximate in value.

Problem 47a: Find the square root of 2268741.

1506.23+. Answer.

Problem 47b: Find the square root of 7596796.

2756.22 +. Answer.

Problem 47c: Find the square root of 101.

10.04987 +. Answer.

$$\sqrt{2} = 1.41421 \qquad \sqrt{10} = 3.16228$$
$$\sqrt{3} = 1.73205 \qquad \sqrt{12} = 3.4661$$
$$\sqrt{6} = \sqrt{2}\sqrt{3} = 2.44949$$

Problem 47d: What is the square root of 96? Find result to three decimal figures.

SQUARE ROOT OF FRACTIONS.

The second power or the square root of a fraction is obtained by squaring the numerator and denominator separately, and writing the result as a fraction. That is, the square root of a fraction is equal to the square root of the numerator divided by the square root of the denominator.

For example, the square root of $\frac{a^2}{b^2}$ is $\frac{a}{b}$ and the square root of $\frac{x^2 + 2xy + y^2}{x^2 - 2xy + y^2} = \frac{x + y}{x - y}$

EXAMPLE 48: Find the square root of $\frac{1}{4}$.

$$\sqrt{1} = 1 \qquad \sqrt{4} = 2$$
$$\therefore \sqrt{\tfrac{1}{4}} = \tfrac{1}{2}$$

PROBLEM 48: Find the square root of $\frac{9}{16}$.

$\frac{3}{4}$. **Answer.**

Problem 48a: Find the square root of $\frac{64}{81}$.

If neither the numerator nor the denominator of a fraction is a perfect square, multiply both numerator and denominator by the denominator, and divide the square root of the new numerator by the original denominator; giving the approximate root.

EXAMPLE 49: **What is the square root of $\frac{7}{4}$?**

$$4 \times \tfrac{7}{4} = \tfrac{28}{4} \qquad \sqrt{28} = 5.2914$$

$$\text{and } \frac{5.2914}{4} = 1.3228. \quad \text{Answer.}$$

PROBLEM 49: What is the square root of $\frac{14}{9}$?

1.2472. Answer.

Problem 49a: What is the square root of $\frac{3}{5}$?

Another method of finding the square root of a fraction is to change the fraction into a decimal, and extract the square root.

EXAMPLE 50: **What is the square root of $\frac{7}{4}$?**

$\frac{7}{4}$ reduced to a decimal, by dividing the numerator by the denominator is 1.75, and the square root of 1.75 is 1.3228. Compare with EXAMPLE 49.

PROBLEM 50: What is the square root of $\frac{7}{8}$?

0.9354. Answer.

Problem 50a: What is the square root of $\frac{5}{3}$?

A few examples and problems in literal expressions will be given as follows:

EXAMPLE 51: **Find the square root of $25a^2x^4$.**

$5ax^2$. Answer.

The proof of the above is to multiply the result by itself. $5ax^2 \times 5ax^2 = 25a^2x^4$. Observe rule on page 34, regarding exponents.

PROBLEM 51: Find the square root of $225a^6b^4x^2$.

$15a^3b^2x$. Answer.

Problem 51a: Find the square root of $81a^8y^4x^2$.

After having found a value for the square root of a number, always multiply the value found, by itself, in order to verify the correctness of the result obtained.

It may be observed that when a monomial* is a perfect square, its coefficient is a perfect square and all the exponents of the literal factors are even numbers.

If a monomial is not a perfect square its approximate square root may be found as follows:

What is the square root of $98ab^4$? Since *the square root of the product of two or more factors is equal to the product of the square roots of the factors,* the given expression may be as follows;

$$\begin{aligned}\sqrt{98ab^4} &= \sqrt{49} \times \sqrt{2} \times \sqrt{a} \times \sqrt{b^4} \\ &= 7 \times \sqrt{2} \times a^{\frac{1}{2}} \times b^2 \\ &= 7 \times 1.414 \times a^{\frac{1}{2}} b^2 \\ &= 9.898\, a^{\frac{1}{2}} b^2\end{aligned}$$

In a similar manner:

$$\sqrt{45a^2b^3c^2d} = \sqrt{9a^2b^2c^2 \times 5bd} = 3abc\sqrt{5bd}$$

RULE FOR EXTRACTING THE SQUARE ROOT OF MONOMIALS.

I. *Find the square root of the coefficient.*†
II. *Divide the exponent of each letter by 2.*

To determine if a given number has any factor that is a perfect square, divide the given number by each of the perfect squares,

4, 9, 16, 25, 36, 49, 64, 81, etc.,

and if it is not divisible without a remainder, it does not contain a factor that is a perfect square.

EXAMPLE 52. A certain general has in command an army of 141376 men. Find the necessary number in "rank and file" to form them into a square.

The necessary number in "rank and file" must be the square root of 141376, or 376 men.

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* See page 40.
† See page 40 for definition of coefficient.

PROBLEM 52: Find the greatest possible number of hills of corn that can be planted on a plot of land comprising a square acre, if the centers of the hills are to be no nearer each other than three and one-half feet.

4165 hills. Answer.

EXAMPLE 53: Divide 100 into two numbers, such that the sum of their square roots will be 14.

Let x denote one number, and $(100 - x)$ denote the other number.

Then by condition;

$$\sqrt{x} + \sqrt{100 - x} = 14$$

and squaring both members gives;

$$x + 2\sqrt{x}\,\sqrt{100 - x} + (100 - x) = 196$$

Which may be reduced as follows:

$$2\sqrt{x}\,\sqrt{100 - x} = 96$$

$$\sqrt{x}\,\sqrt{100 - x} = 48 \text{ (dividing both members by 2)}$$

$$x\,(100 - x) = 2304 \text{ (squaring both members)}$$

$$100x - x^2 = 2304$$

Rearranging and changing signs;

$$x^2 - 100x = -2304$$

Completing the square, by adding to both members the square of one-half the coefficient of x, gives;

$$x^2 - 100x + 2500 = 2500 - 2304$$

$$x - 50 = \pm\sqrt{196} = \pm 14$$

$$x = 50 \pm 14 = 64 \text{ or } 36$$

If $x = 64$ then $100 - x = 36$.
$x = 36$ then $100 - x = 64$.

The square root of $36 = 6$, and the square root of $64 = 8$. $6 + 8 = 14$.

The preceding EXAMPLE 53 may be better comprehended after a study of page 109.

PROBLEM 53: Divide 106 into two numbers, such that the sum of their square roots may be 14.

25 and 81. Answer.

Problem 53a: Divide 10000 into two numbers, such that the sum of their square roots may be 140.

EXAMPLE 54: The sum of a number plus its square root is 90, what is the number?

Let x denote the number, then;

$$x + \sqrt{x} = 90$$

$$x + \sqrt{x} + (\tfrac{1}{2})^2 = 90 + (\tfrac{1}{2})^2 = 90 + \tfrac{1}{4} = \tfrac{361}{4}$$

$$\sqrt{x} + \tfrac{1}{2} = \pm\sqrt{\tfrac{361}{4}} = \pm\tfrac{19}{2}$$

$$\sqrt{x} = -\tfrac{1}{2} \pm \tfrac{19}{2} = 9 \text{ or } -10$$

Therefore, $x = 81$ or 100.

81 is the number required.

LESSON XIX

EXPLAINING EQUATIONS OF THE SECOND DEGREE. QUADRATIC EQUATIONS

Mention was made on page 54 as to the meaning of the *degree* of an equation. It may be said further that if an equation contains two unknown quantities, it is of the second degree when the greatest sum of the exponents of the unknown quantities *in any term,* is equal to 2.

Thus; $x^2 = a$, $ax^2 + bx = c$, and $xy + x = e^2$ are equations of the second degree.

Equations of the second degree may be separated into two classes:

1st. Equations involving only the square of the unknown quantity, and known terms; called *Incomplete Equations.*

2nd. Equations which involve the first and second powers of the unknown quantity, and known terms; called *Complete Equations.*

Examples of incomplete equations are;

$$x^2 + 3x^2 - 4 = 9$$
$$3x^2 - 5x^2 + 5 = a$$

Examples of complete equations are;

$$3x^2 - 5x - 3x^2 + b = c$$
$$2x^2 - 8x^2 - x - a = b$$

Every incomplete equation can be reduced to an equation consisting of two terms, of the general form;

$x^2 = m$, which is sometimes called a *pure quadratic.*

By extracting the square root of both members, the equation becomes;

$$x = \sqrt{m}; \text{ or } x = \pm \sqrt{m}$$

An example illustrating the foregoing will be given.

EXAMPLE 55: What number multiplied by itself will give 6561?

Let x denote the number.
Then $x \times x$ or $x^2 = 6561$.

$$x = \sqrt{6561} = 81. \quad \textbf{Answer.}$$

PROBLEM 55: What number multiplied by itself will give $49a^4b^2x^6$?

$7a^2bx^3$. **Answer.**

Problem 55a: What number, multiplied by itself will produce $1521a^2b^4x^2y^4$?

Problem 55b: What number multiplied by itself will give $625c^4d^4y^8$?

$25c^2d^2y^4$. **Answer.**

A root* of an equation may be defined as any *expression* which substituted for the unknown quantity, will satisfy the conditions of the equation; that is, will render the two members numerically equal to each other.

Thus in the equation $x^2 = 25$,

there are two roots, $+5$ and -5; since either of these numbers substituted for x will satisfy the condition.

Hence it may be stated that;

1st. *Every complete equation of the second degree has two roots.*

2nd. *That these roots are numerically equal, but have unlike signs.*

EXAMPLE 56: What are the roots of the equation $3x^2 + 6 = 4x^2 - 10$?

Rearranging the terms gives:

$$-x^2 = -16$$

And changing all signs; $x^2 = 16$

Therefore; $x = \pm 4$

The roots are $+4$ and -4. Answer.

PROBLEM 56: Find the roots of the equation $\frac{1}{3}x^2 - 8 = \frac{x^2}{9} + 10$.

$x = +9$ and $x = -9$. Answer.

Problem 56a: Find the roots of the equation $4x^2 + 13 - 2x^2 = 45$.

* See page 47.

EXAMPLE 57: Find the numerical value of a number such that if one-third of it is multiplied by one-fourth of it the product shall be 108.

Let x denote the number.

Then by conditions imposed;

$$\frac{x}{3}\times\frac{x}{4} = 108$$

$$\frac{x^2}{12} = 108$$

$$x^2 = 1296$$

$$x = \pm 36$$

PROBLEM 57: Find the number whose square, plus 18, shall be equal to one-half its square plus $30\frac{1}{2}$.

Number $=$ 5. Answer.

EXAMPLE 58: Find the value of two numbers which are to each other as 5 to 6, and the difference of whose squares is 44.

Let x denote the greater number.

Then $\frac{5}{6}x$ will denote the lesser number.

By the conditions;

$$x^2 - \tfrac{25}{36}x^2 = 44$$

$$36x^2 - 25x^2 = 1584$$

$$11x^2 = 1584$$

$$x^2 = 144$$

$$x = 12$$

If $x = 12$, then $\frac{5}{6}$ of $12 = 10$. Therefore one number is 10 and the other is 12.

PROBLEM 58: Find the numerical value of two numbers which are to each other as 3 to 4, and the difference of whose squares is 28.

The two numbers are 6 and 8. Answer.

Problem 58a: Find the numerical value of two numbers which are to each other as 5 to 3, and the difference of whose squares is 400.

Problem 58b: Find the numerical value of two numbers which are to each other as 5 to 11, and the sum of whose square is 584.

The two numbers are 10 and 22. Answer.

Those who intend to enter the engineering profession should very carefully train themselves to solve quadratic equations. It is necessary for Mechanical, Steam, and Electrical engineers, on many occasions, to be able to readily solve equations of the second degree, in connection with the computing of all kinds of efficiency, and questions of power. Especial consideration is therefore given to the principles of second degree equations in this book and students are strongly advised to carefully work out as many equations as possible from other text books; employing as guides the EXAMPLES in this book, and employing as many methods of proving or verifying results, as is possible.

A few general principles relating to quadratics will now be considered.

Suppose a given equation is;

$$x^2 + 4x - 12 = 0$$

Transposing the -12 gives; $x^2 + 4x = 12$. If now 4 be added to both members of the equation, the left hand member becomes a perfect square,* giving;

$$x^2 + 4x + 4 = 12 + 4 = 16$$

Taking the square root of each member gives;

$$x + 2 = \pm \sqrt{16} = \pm 4$$

From which, $x = -2 \pm 4 = +2$ or -6

It should be noted that 4, the number added to both members of the equation, is equal to the square of $\frac{1}{2}$ the coefficient of the term containing x. That is $\frac{1}{2}$ of $4 = 2$, and $2^2 = 4$.

Again suppose the following equation is given;

$$4x^2 + 32x - 132 = 0$$

Transposing the *absolute term* to the right hand side gives:

$$4x^2 + 32x = 132$$

Dividing the coefficient of *each term* by 4, gives;

$$x^2 + 8x = 33$$

Adding to both members the square of one-half the coefficient of x gives;

$$x^2 + 8x + 16 = 33 + 16 = 49$$

Taking the square root of each member gives;

$$x + 4 = \pm 7$$

$$x = -4 \pm 7 = +3 \text{ or } -11$$

* It also happens that the second member is a perfect square in this case.

Substituting the value -11 in the given equation there results:

$$\begin{aligned} 4 \times 121 + 32 \times (-11) &= 132 \\ 484 - 352 &= 132 \\ 132 &= 132 \end{aligned}$$

Substituting the value $x = +3$ gives;

$$\begin{aligned} 4 \times 9 + 32 \times 3 &= 132 \\ 36 + 96 &= 132 \\ 132 &= 132 \end{aligned}$$

A form denoting a perfectly general case may be expressed as follows;

$$ax^2 + bx + c = 0$$

Dividing each member by a gives;

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

If to both members is added the square of one-half the coefficient of x there results;

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{1}{a}\left(\frac{b^2}{4a} - c\right)$$

Expressing the square root of each member gives:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{1}{a}\left(\frac{b^2}{4a} - c\right)}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{1}{a}\left(\frac{b^2}{4a} - c\right)}$$

This may be applied to the equation $4x^2 + 32x - 132 = 0$ as follows;

$$\left.\begin{aligned} a &= 4 \\ b &= 32 \\ c &= -132 \end{aligned}\right\}$$ Substituting these equivalent values in the above expression for x gives;

$$\begin{aligned} x &= -\tfrac{32}{8} \pm \sqrt{\tfrac{1}{4}(\tfrac{1024}{16} + 132)} \\ &= -4 \pm \sqrt{\tfrac{1}{4}(64 + 132)} \\ &= -4 \pm \sqrt{\tfrac{196}{4}} = -4 \pm \sqrt{49} \\ &= -4 \pm 7 = +3 \text{ or } -11. \end{aligned}$$

While *any* equation may be solved by substituting the proper values in the typical or general form, it is far better to treat each equation by itself, applying the following general rule.

1st. *Place the absolute term on the right hand side of the equation.*

2nd. *Divide the coefficient of each term by such a number as to reduce to unity, the coefficient of the term containing the second power of the unknown quantity.*

3rd. *After arranging the terms according to the descending powers of the unknown quantity, change all the signs so as to cause the sign of the term containing the second power to be positive.*

4th. *Add to both members of the equation, the square of one-half the coefficient of the term containing the first power of the unknown quantity.*

5th. *Extract the square root of both members of the equation.*

6th. *Solve for x, by placing x on the left hand side of the equation and all the known or absolute quantities on the right hand side.*

These rules may be expressed differently as follows:

To solve a complete quadratic;

Reduce the equation to the general form $x^2+px=q$, and complete the square by adding to both members the square of one-half the coefficient of x. Extract the square root and solve for x the equation of the first degree thus formed.

The general form in this case will be;

$$x^2+px+\left(\frac{p}{2}\right)^2=\left(\frac{p}{2}\right)^2+q$$

$$x+\frac{p}{2}=\pm\frac{\sqrt{p^2+4q}}{2}$$

$$x=-\tfrac{1}{2}\left(p\pm\sqrt{p^2+4q}\right)$$

Applying this to the equation, $4x^2+32x-132=0$, the following is obtained;

$$x^2+8x=33$$

$p=8$ and $q=33$.

Then; $x=-\frac{1}{2}(8\pm\sqrt{64+132})$
$=-\frac{1}{2}(8\pm\sqrt{196})=-\frac{1}{2}(8\pm 14)$
$=-\frac{1}{2}\times 22$ or $-\frac{1}{2}(-6)=+3$
$=-11$ or $+3$; as found on page 109.

In every case thus far considered, the sign of the term involving the first power of x has been *positive.* It will be well to now consider the case when the sign of this term is *negative.*

Suppose the given equation is of the form;

$$x^2 - px + q = 0$$

Then;
$$x^2 - px = -q.$$

Completing the square gives;

$$x^2 - px + \frac{p^2}{4} = +\frac{p^2}{4} - q.$$

And extracting the square roots;

$$x - \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q} = \pm\sqrt{\frac{p^2 - 4q.}{4}}$$

It should be noted that when the sign of the term containing the first power of x was *positive* the root of the first member was expressed by $x + \frac{p}{2}$ but when this term is negative, the root is expressed by $x - \frac{p}{2}$

Solving for x; $\quad x = \frac{p}{2} \pm \sqrt{\frac{p^2 - 4q}{4}}$

$$= \tfrac{1}{2}(p \pm \sqrt{p^2 - 4q})$$

A careful study of the following examples and problems will enable anyone to readily solve any quadratic found in ordinary practice.

After one has found the squares of a few algebraical expressions, many square root values may be found by a simple *inspection* of the given expressions. Consult pages 46 and 47.

A few typical expressions may be stated as follows;

$(a + b)^2 = a^2 + 2ab + b^2$;	$(a + b)$ = square root of $a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$;	$(a - b)$ = square root of $a^2 - 2ab + b^2$
$(a + 2b)^2 = a^2 + 4ab + 4b^2$;	$(a + 2b)$ = square root of $a^2 + 4ab + 4b^2$
$(5a - 2b)^2 = 25a^2 - 20ab + 4b^2$;	$(5a - 2b)$ = square root of $25a^2 - 20ab + 4b^2$
$(5a + 2b)^2 = 25a^2 + 20ab + 4b^2$;	$(5a + 2b)$ = square root of $25a^2 + 20ab + 4b^2$
$(x + 5)^2 = x^2 + 10x + 25$;	$(x + 5)$ = square root of $x^2 + 10x + 25$
$(x - 5)^2 = x^2 - 10x + 25$;	$(x - 5)$ = square root of $x^2 - 10x + 25$
$(x + y)^2 = x^2 + 2xy + y^2$;	$(x + y)$ = square root of $x^2 + 2xy + y^2$
$(x - y)^2 = x^2 - 2xy + y^2$;	$(x - y)$ = square root of $x^2 - 2xy + y^2$
$(a + \frac{1}{2})^2 = a^2 + a + \frac{1}{4}$;	$(a + \frac{1}{2})$ = square root of $a^2 + a + \frac{1}{4}$
$(a - \frac{1}{4})^2 = a^2 - \frac{1}{2}a + \frac{1}{16}$;	$(a - \frac{1}{4})$ = square root of $a^2 - \frac{1}{2}a + \frac{1}{16}$

See **Problem 10b**: page 48.

EXAMPLE 59: Solve for x; $3ax^2 + 2bx + c = 0$.

Rearranging gives;

$$3ax^2 + 2bx = -c$$

Dividing each term by $3a$ gives;

$$x^2 + \frac{2}{3}\frac{b}{a}x = \frac{c}{3a}$$

Completing the square by adding the square of $\frac{1}{2}$ the coefficient of x to both members of the equation gives;

$$x^2+\tfrac{2}{3}\frac{b}{a}x+\left(\frac{2b}{2\times3a}\right)^2=-\frac{c}{3a}+\left(\frac{2b}{2\times3a}\right)^2$$

$$x^2+\tfrac{2}{3}\frac{b}{a}x+\left(\frac{b}{3a}\right)^2=-\frac{c}{3a}+\frac{b^2}{9a^2}$$

$$=\frac{-3ac+b^2}{9a^2}=\frac{b^2-3ac}{9a^2}$$

Extracting the square root gives;

$$x+\frac{b}{3a}=\pm\sqrt{\frac{b^2-3ac}{9a^2}}=\pm\frac{\sqrt{b^2-3ac}}{3a}$$

$$x=-\frac{b}{3a}\pm\frac{\sqrt{b^2-3ac}}{3a}$$

$$=\frac{-b+\sqrt{b^2-3ac}}{3a}\text{ or }\frac{-b-\sqrt{b^2-3ac}}{3a}$$

This example may also be solved by substituting the proper values in the general equation given on page 111.

In this case $p=\frac{2}{3}\frac{b}{a}$, and $q=-\frac{c}{3a}$

SIMPLE QUADRATICS

EXAMPLE 60: Solve the following equation to obtain the value of *x*.

$$\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$$

The above is the same as;

$$\frac{2\sqrt{x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$$

Multiplying both members by $\sqrt{x}$ gives;

$\dfrac{2x+2\sqrt{x}}{4+\sqrt{x}}=4-\sqrt{x}$, and multiplying both members by $4-\sqrt{x}$ gives;

$2x+2\sqrt{x}=16-x$, which combines to give;

$3x+2\sqrt{x}=16$. Dividing each term by 3, gives;

$x+\frac{2}{3}\sqrt{x}=\frac{16}{3}$ Completing the square by adding to both members the square of $\frac{1}{2}$ the coefficient of $\sqrt{x}$ gives;

$$x+\tfrac{2}{3}\sqrt{x}+(\tfrac{1}{3})^2=\tfrac{16}{3}+(\tfrac{1}{3})^2=\tfrac{16}{3}+\tfrac{1}{9}=\tfrac{48}{9}+\tfrac{1}{9}=\tfrac{49}{9}$$

Extracting the square root of both members gives;

$$\sqrt{x}+\tfrac{1}{3}=\pm\sqrt{\tfrac{49}{9}}=\pm\tfrac{7}{3}$$

$$\sqrt{x}=-\tfrac{1}{3}\pm\tfrac{7}{3}=\tfrac{6}{3}\text{ or }-\tfrac{8}{3}$$

From which $x=(\tfrac{6}{3})^2$ or $(-\tfrac{8}{3})^2=\tfrac{36}{9}$ or $\tfrac{64}{9}=4$ or $\tfrac{64}{9}$

EXAMPLE 61. Solve the following equation to obtain an expression for *x*. In short, solve for *x*.

$$\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}$$

Reduce the second member to a common denominator, giving;

$$\frac{1}{a+b+x}=\frac{bx+ax+ab}{abx}$$

Multiplying both members by abx gives;

$$\frac{abx}{a+b+x}=bx+ax+ab$$

Multiplying both members by $a+b+x$, gives;

$$abx=(a+b+x)\ (bx+ax+ab)$$
$$=a^2b+a^2x+3abx+ab^2+b^2x+ax^2+bx^2$$

From which is obtained;

$$ax^2+bx^2+a^2x+b^2x+2abx+a^2b+ab^2=0$$
$$(a+b)x^2+(a^2+2ab+b^2)x=-a^2b-ab^2$$

Dividing both members by $(a+b)$ gives;

$$x^2+(a+b)x=\frac{-a^2b-ab^2}{a+b}=\frac{-ab(a+b)}{a+b}=-ab$$

Adding the square of $\frac{1}{2}$ the coefficient of x to both members gives:

$$x^2+(a+b)x\ \frac{(a+b)^2}{4}=\frac{(a+b)^2}{4}-ab=\frac{(a+b)^2-4ab}{4}$$

Extracting the square root of both members, gives;

$$x+\frac{a+b}{2}=\pm\frac{a-b}{2}$$

From which $x=-\dfrac{a+b}{2}\pm\dfrac{a-b}{2}=\dfrac{-a-b+a-b}{2}=-b$ or $-a$

EXAMPLE 62: A square picture is surrounded by a frame. The side of the picture exceeds by an inch the width of the frame, and the number of square inches in the frame exceeds by 124 the number of inches in the perimeter of the picture. Find the area of the picture, and the width of the frame.

Let x denote the area of the picture.
Let one side of the picture be denoted by $\sqrt{x}$.
Let $\sqrt{x}-1$ denote the width of the frame.

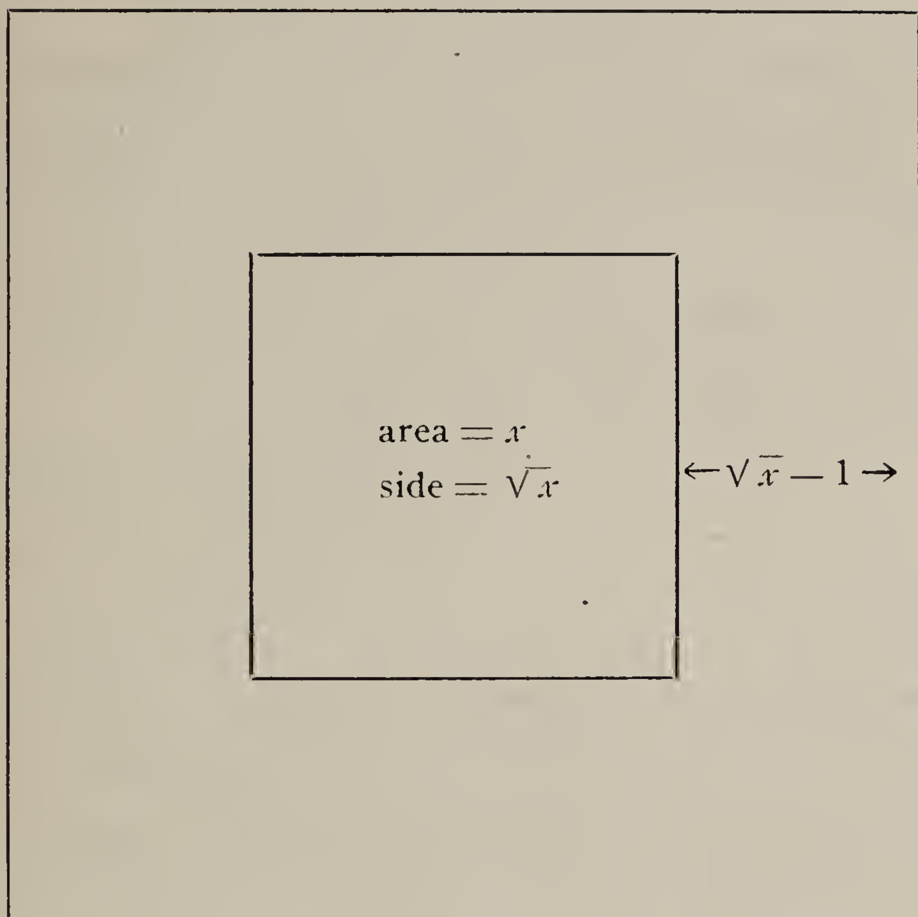

FIG. 2

$[2(\sqrt{x}-1)+\sqrt{x}]^2-x=$ area of frame.

Then;

$$[2(\sqrt{x}-1)+\sqrt{x}]^2-x-124=4\sqrt{x}$$
$$(3\sqrt{x}-2)^2-x-124=4\sqrt{x}$$
$$9x-12\sqrt{x}+4-x-124=4\sqrt{x}$$
$$8x-16\sqrt{x}=120$$
$$x-2\sqrt{x}=15$$
$$x-2\sqrt{x}+(1)^2=15+(1)^2=16$$
$$\sqrt{x}-1=\pm 4$$
$$\sqrt{x}=5 \text{ or } -3$$
$x=25$ or nine. Answer.

EXAMPLE 63: The circumference of the front-wheel of a carriage is less by 4 feet than that of the back-wheel. In travelling 1200 feet, the front-wheel makes 25 revolutions more than the back-wheel. Find the circumference of each wheel.

Let x denote the circumference of the back-wheel.

Let $x - 4$ denote the circumference of the front-wheel.

Then $\frac{1200}{x}$ denotes the number of revolutions of the back-wheel

and $\frac{1200}{x-4}$ denotes the number of revolutions of the front-wheel.

Then; $\frac{1200}{x} = \frac{1200}{x-4} - 25$

$$1200x - 4800 = 1200x + 25x^2 - 100x$$
$$25x^2 - 100x = 4800$$
$$x^2 - 4x = 192$$
$$x^2 - 4x + 4 = 196$$
$$x - 2 = \pm 14$$
$$x = 16 \text{ or } -12$$

Circumference of back-wheel = 16 feet,

Circumference of front-wheel = 16 — 4 = 12 feet. Answer,

The *diameter* of either wheel may be found by dividing each circumference by 3.1415.

LESSON XX

SIMPLE QUADRATIC EQUATIONS

EXAMPLE 64: A and B gained \$2100 in trade. A's money was in the firm during 15 months, and he received in principal and gain, \$3900. B's money, which was \$5000, was in the firm 12 months. How much money did A put into the firm?

Let x denote A's principal put into the firm.

Then $3900 - x$ denotes A's gain; while $2100 - (3900 - x)$ denotes B's gain.

$$\frac{3900 - x}{2100 - 3900 + x} = \frac{5}{4} \times \frac{x}{5000} = \frac{x}{4000}$$

$$15{,}600{,}000 - 4000x = 2100x - 3900x + x^2$$

$$x^2 + 2200x = 15{,}600{,}000$$

$$x^2 + 2200x + \overline{1100}^2 = 15{,}600{,}000 + 1{,}210{,}000$$

$$x + 1100 = \pm 4100$$

$x = \pm 4100 - 1100 = \$3000$. **Answer.**

A's gain = \$900. B's gain = \$1200. While B's capital was in the firm during a shorter time, he received \$300 more than A did, because of the larger amount invested.

EXAMPLE 65: The sum of \$120 was divided between a certain number of persons. If each person had received \$7 less, he would have received as many dollars as there were persons. Find the number of persons.

Let x denote the number of persons.

Then $\frac{120}{x}$ denotes number of dollars each received.

By the condition imposed;

$$\frac{120}{x} - 7 = x$$

$$120 - 7x = x^2$$

$$x^2 + 7x = 120$$

$$x^2 + 7x + (\tfrac{7}{2})^2 = 120 + (\tfrac{7}{2})^2 = \tfrac{529}{4}$$

$$x + \tfrac{7}{2} = \pm \tfrac{23}{2}$$

$x = \pm \frac{23}{2} - \frac{7}{2} = 8$ (or -15). **Answer.**

EXAMPLE 66: A person's income is $5000. After deducting a percentage for income tax, and then a percentage, less by one than that of the income tax, from the remainder, his income is reduced to $4656. What is the rate per cent of the income tax?

Let x denote the rate of the income tax.

Then; $5000 \times \frac{x}{100}$ denotes the amount of income tax.

$5000 - \frac{5000}{100}x$ denotes the remainder after deducting the income tax.

$(5000 - 50x)\frac{x-1}{100}$ denotes extra tax on remainder.

Then by conditions of the statement;

$$5000 - 50x - (5000 - 50x)\frac{x-1}{100} = 4656$$

$$5000 - 50x - \left(50x - 50 - \frac{x^2}{2} + \frac{x}{2}\right) = 4656$$

$$5000 - 50x - 50x + 50 + \frac{x^2}{2} - \frac{x}{2} = 4656$$

$$\frac{x^2}{2} - \frac{201}{2}x = -394$$

$$x^2 - 201x = -788$$

$$x^2 - 201x + \left(\frac{201}{2}\right)^2 = -788 + \left(\frac{201}{2}\right)^2 = +\frac{17149}{4}$$

$$x - \frac{201}{2} = \pm\frac{193}{2}$$

$$x = \frac{201}{2} \pm \frac{193}{2} = 197 \text{ or } 4. \quad \textbf{Answer.}$$

The rate could not be over 100, therefore the value 4 is correct.

EXAMPLE 67: If $2000 amounts to $2205 when put at compound interest for 2 years, the interest being compounded annually, what is the rate per cent per annum.

Let x denote the rate of interest, in per cent.
Then the amount at the end of the first year is expressed by;

$$2000 + \left(\frac{x}{100} \times 2000\right)$$

The amount at the end of two years is expressed by;

$$\left[2000 + \left(\frac{x}{100} \times 2000\right)\right] + \left[2000 + \left(\frac{x}{100} \times 2000\right)\frac{x}{100}\right] = 2205$$

This reduces to;

$$2000 + 20x + 20x + \frac{x^2}{5} = 2205$$

$$\frac{x^2}{5} + 40x = 205$$

$$x^2 + 200x = 1025$$

$$x^2 + 200x + \overline{100}^2 = 1025 + 10000 = 11025$$

$$x + 100 = \pm \sqrt{11025} = \pm 105$$

$x = -100 + 105 = 5$ per cent. Answer.

EXAMPLE 68. A man travelled 105 miles. Had he gone 4 miles more in an hour, he would have completed the journey in $9\frac{1}{3}$ hours less than he did. How many miles an hour did he travel?

Let x denote his rate of travel in miles per hour.

Then according to the condition;

$\frac{105}{x} = \frac{105}{x+4} + 9\frac{1}{3}$. From which is obtained;

$\frac{105}{x} = \frac{105}{x+4} + \frac{28}{3}$. This reduces to;

$$\frac{105}{x} = \frac{315 + 28x + 112}{3x + 12}$$

$$315x + 1260 = 315x + 28x^2 + 112x$$

$$28x^2 + 112x = 1260$$

$$x^2 + 4x = 45$$

$$x^2 + 4x + (2)^2 = 45 + 4 = 49$$

$$x + 2 = \pm 7$$

$x = -2 \pm 7 = 5$ miles per hour. Answer.

EXAMPLE 69: A crew can row a boat 8 miles down stream and back again in $4\frac{4}{5}$ hours; if the rate of the stream is 4 miles an hour, find the rate at which the crew can row the boat in still water.

The solution is based on the fact that in general a rate $= \frac{\text{distance}}{\text{time}}$ That is; that the rate of any moving body is equal to the distance passed over by the body, divided by the time required in passing; *assuming uniform motion.* This is expressed in symbols by;

$$\text{rate} = \frac{d}{t}$$

Let x denote the rate in still water.

Then the rate going down stream $= (x+4)$, and the rate going up stream is $(x-4)$. The distance going down is 8 miles, and is the same going up.

Let t denote the time in going down and t' the time in going up. Then;

$$t = \frac{8}{x+4} \text{ and } t' = \frac{8}{x-4}$$ But the total time is

$$t + t' = 4\tfrac{4}{5} = \tfrac{24}{5} \text{ hours.}$$

Then $$t + t' = \frac{8}{x+4} + \frac{8}{x-4} = \tfrac{24}{5}$$

Or; $$8x - 32 + 8x + 32 = \tfrac{24}{5}(x^2 - 16)$$ $x^2-16=(x+4)(x-4)$

$$80x = 24(x^2 - 16)$$

Rearranging gives; $x^2 - \frac{10}{3}x = 16$

Completing the square; $x^2 - \frac{10}{3}x + (\frac{10}{6})^2 = 16 + \frac{100}{36} = \frac{676}{36}$

Whence $x = -\frac{16}{6}$ or $\frac{36}{6} = 6$ **miles per hour. Answer.**

EXAMPLE 70: A rectangular garden is surrounded by a walk 7 feet wide; the area of the garden is 15,000 square feet, and of the walk 3696 square feet. Find the length and breadth of the garden.

Figure 3 shows the arrangement.

Let x denote the length of the outer rectangle. Then $\frac{15000+3696}{x}$ or $\frac{18696}{x}$ denotes the breadth of the outer rectangle.

The length of the garden is therefore denoted by $(x-14)$, and the breadth of the garden by $\frac{15000}{x-14}$

The area of the garden is equal to its length times its breadth, and is 15000 sq. feet.

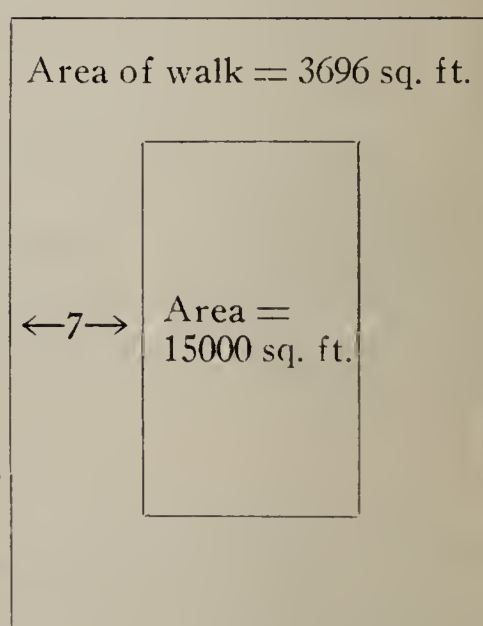

FIG. 3

Therefore; $$\frac{15000}{x-14} = \frac{18696}{x} - 14$$

From which;

$$15000x - 18696x - 14 \times 18696 = 14x^2 + 196x$$

Which reduces to;

$$14x^2 - 3892x = -14 \times 18696$$
$$x^2 - 278x = -18696$$
$$x^2 - 278x + (139)^2 = -18696 + (139)^2 = 625$$
$$x - 139 = \pm\sqrt{625} = \pm 25$$
$$x = 139 \pm 25$$
$$= 164 \text{ or } 114$$

The length of the garden $= 164 - 14 = 150$ feet.

The breadth of the garden $= \dfrac{15000}{150} = 100$ feet.

EXAMPLE 71: A merchant sold a quantity of wheat for \$56, and gained as many per cent as the wheat cost dollars. What was the cost of the wheat?

Let x denote the cost of the wheat, in dollars.

One per cent is therefore denoted by $\frac{x}{100}$; if the gain was x per cent, the gain would be denoted by $\frac{x^2}{100}$ ($= x \times \frac{x}{100}$). The gain added to the cost will give the selling price.

The condition is therefore expressed by;

$$x + \tfrac{x^2}{100} = 56$$
$$x^2 + 100x = 5600 \qquad \text{multiplying each term by 100}$$

Completing the square;

$$x^2 + 100x + (50)^2 = 5600 + (50)^2 = 8100$$

Extracting roots; $\quad x + 50 = \pm 90$

Cost of the wheat was; $x = -50 \pm 90 = 40$ **dollars. Answer.**

This may be verified as follows;
The wheat cost \$40. 40% of \$40 = \$16. \$ 40 + \$16 = \$56.

EXAMPLE 72: A farmer bought a number of sheep for \$378. Having lost 6, he sold the remainder for \$10 a head more than they cost him, and gained \$42. How many did he buy?

Let x denote the number of sheep purchased.

Then $\dfrac{378}{x} =$ cost in dollars for each sheep.

$$\left(\frac{378}{x} + 10\right)(x - 6) = \$378 + \$42$$

$378 + 10x - \dfrac{2268}{x} - 60 = \$420.$ Multiplying by x, gives;

$318x + 10x^2 - 2268 = 420x$

or $10x^2 - 102x = 2268.$ Dividing by 10 gives;

$x^2 - \frac{102}{10}x = \frac{2268}{10}$

Completing the square gives;

$x^2 - \frac{102}{10}x + (\frac{51}{10})^2 = \frac{2268}{101} + (\frac{51}{10})^2 = \frac{25281}{100}$

Extracting the roots;

$x - \frac{51}{10} = \pm \frac{159}{10}$

$x = \frac{51}{10} \pm \frac{159}{10} = \frac{210}{10}$ $= 21$ **sheep. Answer.**

It is obvious that the *converse* of the preceding Example would be the actual practical reality. The farmer would know how many he purchased at first, and the number lost, and selling the remainder at a certain price, would realize a certain sum; either gaining or losing.

EXAMPLE 73: Find a number such that the sum of its cube, twice its square, and the number itself, is twenty times the next higher number.

Let x denote the number, then the condition stated is expressed by;

$x^3 + 2x^2 + x = 20\,(x + 1)$ (The next higher number is the given number plus *one*.)

$x^3 + 2x^2 + x = 20x + 20$

From which;

$x^3 + 2x^2 - 19x - 20 = 0$

One method of solving such an equation will be to substitute different numerical values for x until one is found that satisfies the equation. In applying this method it will be logical to begin with 1 and continue with the positive integers in order, until one is found that satisfies the equation.

Substituting 1; gives $1 + 2 - 19 - 20 = 0$
or $3 - 39 = 0$; which is evidently not correct

Substituting 2; gives $8 + 8 - 38 - 20 = 0$
or $16 - 58 = 0$; also not correct.

Substituting 3; gives $27 + 18 - 57 - 20 = 0$
or $45 - 77 = 0$; not correct.

It may be now noted that as the numbers substituted, increase in value, the more nearly the required condition is fulfilled.

Substituting 4; gives $64 + 32 - 76 - 20 = 0$

or $96 - 96 = 0$; $96 = 96$ which *does* fulfill the condition. x is therefore $= 4$.

If $x = 4$ then $x - 4 = 0$, and $(x - 4)$ is one of the factors of
$x^3 + 2x^2 - 19x - 20 = 0$

Dividing $x^3 + 2x^2 - 19x - 20 = 0$ by $(x - 4)$ gives $x^2 + 6x + 5$, the factors of which may be seen by inspection to be $(x + 1)$ and $(x + 5)$.

It is evident therefore that;

$$(x - 4)\ (x + 1)\ (x + 5) = 0$$

Dividing both members by $(x - 4)\ (x + 1)$ gives;

$$x + 5 = 0$$

from which; $x = -5$; another value that fulfills the condition.

Also; $x + 1 = 0$
$x = -1$; a third correct value.

It is obvious that *negative* numbers could have been substituted in the original equation as well as the positive integers. Then the value -1 could have been substituted and then; $-1 + 2 - 1 = 0$, or $2 = 2$ would have resulted at once.

It is further evident that fractional values may always be substituted as well as integers; and *negative* fractions as well as positive ones.

EXAMPLE 74: A man bought a horse for a certain sum, and after a time sold the horse for $24. By the sale the man loses as much per cent upon the purchase price as the horse cost him. What did the man pay for the horse?

Let x denote the purchase price in dollars.
Then $x - 24$ expresses the loss in dollars.

Since the loss was x per cent, the loss was $\frac{x}{100}$ on each dollar; upon x dollars the loss would be $x \times \frac{x}{100} = \frac{x^2}{100}$

The following will then express the condition;

$$\frac{x^2}{100} = x - 24$$

$$x^2 = 100x - 2400$$
$$x^2 - 100x = -2400$$
$$x^2 - 100x + (50)^2 = 2500 - 2400$$
$$x - 50 = \pm\sqrt{100} = \pm 10$$
$$x = \$60 \text{ or } \$40$$

Both values satisfy the conditions.

EXAMPLE 75: Find four consecutive numbers such that if the first two be taken as the digits of a number, that number is equal to the product of the other two.

Let the four numbers be denoted by x

$$x + 1$$
$$x + 2$$
$$x + 3$$

Then according to the conditions;

$$10x + (x + 1) = (x + 2)\ (x + 3)$$
$$10x + x + 1 = x^2 + 5x + 6$$
$$-x^2 + 6x = 5$$
$$x^2 - 6x = -5$$
$$x^2 - 6x + 9 = -5 + 9 = 4$$

Extracting roots; $x - 3 = \pm\sqrt{4} = \pm 2$

$x = 3 \pm 2 = \mathbf{5}$ **or 1. Answer.**

The first number is therefore 5 or 1.

If the first number is 5 the others are **6, 7, and 8. Answer.**

If the first two are taken as the digits of a number the number is 56, and $56 = 7 \times 8$.

If the first number is 1 the others are 2, 3, and 4.

The first two as digits give 12 and $12 = 3 \times 4$.

This example shows the advisability of retaining the *negative* value of the root numbers as well as the positive values. Using both values in this case gives two correct answers.

EXAMPLE 76: A man in mowing his lawn, starts on the outside of the lawn and goes round and round the lawn, which is a rectangle 100 feet by 120 feet. What width will he have to cut to be $\frac{1}{3}$ done? To be $\frac{2}{3}$ done? To be $\frac{1}{2}$ done?

Area of the lawn $= 120 \times 100 = 12000$ square feet.

$\frac{1}{3}$ of the area $= 4000$ sq. feet.

Let x denote the *width* cut over; consult figure 4.

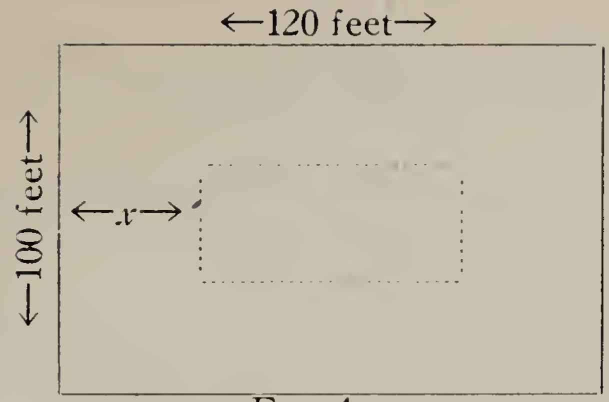

FIG. 4

Then; $(100-2x)\ (120-2x) = 8000$

$$12000 - 440x + 4x^2 = 8000 = \tfrac{2}{3} \text{ of } 12000$$
$$4x^2 - 440x = -4000$$
$$x^2 - 110x = -1000$$
$$x^2 - 110x + (55)^2 = (55)^2 - 1000 = 3025 - 1000 = 2025$$
$$x - 55 = 55 \pm \sqrt{2025}$$
$$x = 55 \pm 45 = \textbf{100 or 10 feet.} \quad \textbf{Answer.}$$

Adopting the value 100 will mean the complete lawn is mown, which is contrary to the hypothesis. The value of 10 feet is the correct value to be selected in the given case.

The correctness of the result may be checked as follows;

$120 \times 10 = 1200$; $1200 \times 2 = 2400$ $(100-20)\ 10 = 800$

$800 \times 2 = 1600$; $2400 + 1600 = 4000$

4000 square feet is $\frac{1}{3}$ of total area of 12000 square feet.

Second Part

To find the width when $\frac{2}{3}$ of the lawn has been mown; let x_1 denote the *width cut over* in this case.

Then by condition;

$$(100-2x_1)\ (120-2x_1) = 4000 = \tfrac{1}{3} \text{ of total area.}$$
$$12000 - 440x_1 + 4x_1^2 = 4000$$
$$4x_1^2 - 440x_1 = -8000$$
$$x_1^2 - 110x_1 = -2000$$
$$x_1^2 - 110x_1 + (55)^2 = (55)^2 - 2000 = 3025 - 2000 = 1025$$
$$x_1 - 55 = \pm\sqrt{1025}$$
$$x_1 = 55 \pm 32.015$$
$$= \textbf{87.015 or 22.985.} \quad \textbf{Answer.}$$

It is evident that the value 87.015 cannot logically fulfill the conditions; since $2 \times 87 = 174$, which is greater even than the *length* (120 feet) of the lawn.

Employing 22.985 as the proper value, the following must be true;

$100 - 2 \times 22.985 = 100 - 45.970 = 54.03$

$120 - 2 \times 22.985 = 120 - 45.970 = 74.03$

$54.03 \times 74.03 = 3999.84$ square feet. This is the *approximate* value, which may be increased to a value nearer to 4000, the true value, by carrying out the results to more decimal figures.

When the lawn is $\frac{1}{2}$ cut over, the condition is expressed by;

$(100 - 2x)(120 - 2x) = 6000$; From which $x = 16$ feet.

EXAMPLE 77: The electric light poles along a certain road are set at equal intervals. If there were two more in each mile, the interval between the poles would be decreased by 20 feet. Find the number of poles per mile.

Let x denote number of poles per mile.

Then $\frac{5280}{x}$ denotes the distance, in feet, between poles.

By the given conditions the following must be true;

$$\frac{5280}{x+2} = \frac{5280}{x} - 20$$

$$5280x = 5280x + 10560 - 20x^2 - 40x$$

$$20x^2 + 40x = 10560$$

$$x^2 + 2x = 528$$

$$x^2 + 2x + (1)^2 = 528 + (1)^2$$

$$x + 1 = \pm\sqrt{529} = \pm 23$$

$$x = 22 \text{ or } -24$$

There are 22 poles per mile. In this case the distance between poles is $\frac{5280}{22} = 240$ feet.

If there were 24 poles per mile, the distance between poles would be $\frac{5280}{24} = 220$ feet; which is 20 feet less than 240 feet.

Such a problem is of no particular value except insofar as it furnishes a certain amount of mental training.

The practical condition would be to find the saving effected by using 22 poles in place of the 24, if the cost of the poles set in position were \$7.00 per pole.

Decreasing the number of poles per miles will effect a saving in poles, insulators and cross arms, but allows danger of breaking of wire due to wind and sleet acting on the longer spans of wire.

SIMULTANEOUS QUADRATIC EQUATIONS

EXAMPLE 78: Solve the following for x and y.

$$8x^2 - 3xy - y^2 = 40$$
$$9x^2 + xy + 2y^2 = 60$$

Multiply the first equation by 3 and the second by 2, then;

$$24x^2 - 9xy - 3y^2 = 120$$
$$18x^2 + 2xy + 4y^2 = 120$$

$24x^2 - 9xy - 3y^2 = 18x^2 + 2xy + 4y^2$. Which properly combined reduces to; $6x^2 - 11xy - 7y^2 = 0$. The factors of the last equation are; $(3x - 7y)\ (2x + y) = 0$

Then;

$$2x = -y \quad \text{and} \quad 3x = 7y$$
$$x = -\tfrac{y}{2} \qquad\qquad x = \tfrac{7}{3}y$$

Substituting these values in either one of the original equations gives;

$$8\tfrac{y^2}{4} + 3\tfrac{y}{2}y - y^2 = 40$$
$$2y^2 + \tfrac{3}{2}y^2 - y^2 = 40$$
$$(1 + \tfrac{3}{2})y = 40$$
$$\tfrac{5}{2}y^2. = 40$$
$$y^2 = \tfrac{80}{5} = 16$$
$$y = \pm 4; \quad \text{therefore } x = \mp 2$$
$$\text{Also } y = \pm\tfrac{6}{7} \text{ and } x = \mp\tfrac{6}{11}$$

EXAMPLE 79: Solve the following equations for x and y.

$$2x^2 + 3xy + 12 = 3y^2$$
$$3x + 5y + 1 = 0$$

The value of x from the second equation is;

$$x = \frac{-(5y + 1)}{3} \text{ from which } x^2 = \frac{25y^2 + 10y + 1}{9}$$

Substituting these values in the first equation gives;

$$2\left(\frac{25y^2 + 10y + 1}{9}\right) + 3y\left(\frac{-5y + 1}{3}\right) + 12 = 3y^2$$

From which is obtained;

$$\frac{50y^2 + 20y + 2}{9} - \frac{3y + 15y^2}{3} - 3y^2 = -12$$

$$50y^2 + 20y + 2 - 9y - 45y^2 - 27y^2 = -108$$

$$-22y^2 + 11y = -110$$

$$y^2 - \tfrac{1}{2}y = 5$$

Completing the square gives;

$$y^2 - \tfrac{1}{2}y + \tfrac{1}{16} = 5 + \tfrac{1}{16}$$

$$y - \tfrac{1}{4} = \pm\sqrt{\tfrac{81}{16}} = \pm\tfrac{9}{4}$$

From which; $y = \frac{5}{2}$ or -2

Therefore; $x = -\frac{9}{2}$ or 3

LESSON XXI

RATIO AND PROPORTION

The fundamental definitions pertaining to *ratio* and *proportion* are given in LESSON XI, page 50, but since "ratio and proportion" really play a very important rôle in our daily existence, it will be well to consider the matter more at length.

It is well to keep in mind that *a ratio is a quotient* arising from dividing one quantity by another quantity of the same kind.

A ratio is not a proportion.

Proportion: A proportion is an expression of the equality of two ratios.

A ratio is commonly expressed as $4:10$; or $3:9$; in each of the two ratios expressed, the first figure is called the **antecedent,** and the second figure is called the **consequent.** The antecedent and the consequent together are called the **terms** of a ratio.

Rule: *Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.*

For example; $\frac{3}{4}=\dfrac{2\times3}{2\times4}=\frac{6}{8}=\dfrac{\frac{6}{2}}{\frac{8}{2}}=\frac{3}{4}$

The following are equivalent expressions;

$$4\div10=\tfrac{4}{10}=4:10=\frac{\text{dividend}}{\text{divisor}}=\frac{\text{numerator}}{\text{denominator}}=\frac{\text{antecedent}}{\text{consequent}}$$

These might be arranged differently as follows:

$$4\div10=\frac{\text{dividend}}{\text{divisor}}$$

$$\tfrac{4}{10}=\frac{\text{numerator}}{\text{denominator}}$$

$$4:10=\frac{\text{antecedent}}{\text{consequent}}$$

One very important relation pertaining to ratios should be kept clearly in mind; the relation of one number to another of the same kind expressed by the quotient of the first divided by the second, is called the ratio of the first to the second.

For example; the ratio of 2 to 4 is $\frac{2}{4}$ $(=\frac{1}{2})$ or $2:4$, not $4:2$ or $\frac{4}{2}$ $(=2)$. It is evident that the ratio of 2 to 4 is very different from the ratio of 4 to 2.

Rule: *In any proportion the product of the means is equal to the product of the extremes.*

This rule is important, and its application should be carefully understood.

The following will illustrate a few applications of the rule. Given the proportions;

$4:10=20:50$, then $4 \times 50=10 \times 20$.
$3:9=6:18$, then $3 \times 18=9 \times 6$.
$x:y=a:b$, then $bx=ay$.
$2a:3b=4c:6d$, then $2 \times 6 \times a \times d=3 \times 4 \times b \times c$.

The fundamental theory of the above rule may be explained as follows:

The proportion $4:10=20:50$ may be written $\frac{4}{10}=\frac{20}{50}$

Suppose both members are first multiplied by 10;

$$\text{then } \frac{4 \times 10}{10}=\frac{200}{50} \text{ or } 4=\frac{200}{50}$$

Next suppose both members of the last equation be multiplied by 50;

$$\text{then } 4 \times 50=\frac{200 \times 50}{50} \text{ or } 200=200$$

This proves that the product of the means is equal to the product of the extremes.

EXAMPLE 80: What is the ratio of 3 days to 4 weeks?

Since a ratio is the quotient of similar quantities, weeks must be reduced to *days*. 4 weeks $=4 \times 7=28$ days. Therefore the ratio of 3 days to 4 weeks may be expressed by $\frac{3}{28}$ or by **3: 28. Answer.**

PROBLEM 80: What is the ratio of one inch to one foot?

One inch is $\frac{1}{12}$ of one foot. $\frac{1}{12}$ **or 1: 12. Answer.**

EXAMPLE 81: Find two numbers in the ratio of 16 to 9 such that, if each number be diminished by 8, they will be in the ratio of 12 to 5.

Let x denote one number, and y denote the other number.

Then $\frac{x}{y}=\frac{16}{9}$; or $9x=16y$. From which $x=\frac{16}{9}y$ or $y=\frac{9}{16}x$

From the second condition the following is true;

$$\frac{x-8}{y-8}=\frac{12}{5}=5x-40=12y-96=5x-12y=-56$$

By substituting in the last equation, the value of $y=\frac{9}{16}x$, the following results; $5x-\frac{108}{16}x=-56$

$$\frac{80}{16}x-\frac{108}{16}x=-56$$

$$-\frac{28}{16}x=-56$$

$$x=\frac{16}{28}\times 56=\frac{4}{7}56=\frac{224}{7}=32$$

Therefore one number is **32. Answer.** Substituting this value in the first equation gives; $\frac{32}{y}=\frac{16}{9}$ from which $y=\frac{9}{16}32=\frac{288}{16}=$ **18.**

Answer.

PROBLEM 81: Find two numbers in the ratio of 5 to 4 such that, if each number be diminished by 4, they will be the ratio of 2 to 3.

$x=\frac{20}{7}$ and $y=\frac{16}{7}$. **Answer.**

Problem 81a: Find two numbers in the ratio of 4 to 5 such that, if each number be diminished by 4, they shall be in the ratio of 2 to 3.

EXAMPLE 82: Divide 36 into two parts such that the greater diminished by 4 shall be to the lesser increased by 3, as 3 is to 2.

Let x denote the greater part.

Then $36-x$ denotes the lesser part.

By condition;

$$\frac{x-4}{(36-x)+3}=\frac{3}{2}$$

Which may be reduced as follows;

$$2x-8=108-3x+9=117-3x$$

$$5x=125$$

$x=25=$ **greater part. Answer.**

$36-25=11=$ **lesser part. Answer.**

PROBLEM 82: Divided 575 into two parts such that the greater diminished by 25 shall be to the lesser increased by 50, as 11 is to 13.

Greater = 300; lesser = 275. Answer.

Problem 82a: Divide 575 into two parts such that the greater diminished by 25 shall be to the lesser increased by 50, as 13 is to 9.

LESSON XXII

SERIES; ARITHMETICAL PROGRESSIONS

Series: An arrangement of numbers, succeeding each other according to some fixed law, is called a *series.*

Terms of a Series: The successive numbers form the *terms* of a series.

A series may be denoted by symbols and letters as well as by numbers.

Two quantities of the same kind may be compared with each other in two ways:—

1st. By considering *how much* greater or less one is than the other; indicated by their difference.

2nd. By considering *how many times* one is greater or less than the other, which is shown by their quotient.

By comparing the numbers 4 and 16 with respect to their difference, it is seen that 16 *exceeds* 4 by 12, and in comparing them with reference to their quotient, it is seen that 16 *contains* 4, four times; or that 16 is four times as great as 4.

The first of these methods is called *Arithmetical Progression,* and the second is called *Geometrical Progression.*

Hence, *Arithmetical Progression considers the relation of quantities with respect to their difference, and Geometrical Progression the relation of quantities with respect to their quotient.*

Arithmetical Progression: An arithmetical progression may be defined as a series, each term of which (*except the first*), is formed from the preceding term by the addition or subtraction of a constant number.

The constant number which is added, is called the *common difference.*

The first term of an arithmetical progression is called an *antecedent,* and the second term a *consequent.* Any number of an arithmetical progression may be considered as an antecedent, and its succeeding number or term, a consequent.

Given the five numbers;

$$2,\ 4,\ 6,\ 8, \text{ and } 10.$$

The difference between the first and second, and between the second and third, etc., is 2. That is, the common difference is 2. By adding 2 to the last term given, another term can be derived and the process continued.

Two other progressions are given as follows;

$$1, \quad 4, \quad 7, \quad 10, \quad 13, \quad 16, \quad 19, \quad 22$$
$$60, \quad 56, \quad 52, \quad 48, \quad 44, \quad 40, \quad 36, \quad 32.$$

The first of these may be called an *increasing progression* or an *ascending progression,* while the second is called a *decreasing* or *descending progression.*

In the first the common difference is 3, while in the second the common difference is -4.

The *natural series of numbers* 1, 2, 3, 4, 5, 6, 7, 8, 9 .... constitute an arithmetical progression. The common difference is unity.

An example of an ascending progression expressed algebraically would be;

$$a, \ a+d, \ a+2d, \ a+3d, \ a+4d,$$

and of a descending one;

$$a, \ a-d, \ a-2d, \ a-3d, \ a-4d.$$

See also, series at top of page 38.

The consideration of the principles of arithmetical progression may be made general by considering the following;

Let a, $a+d$, $a+2d$, $a+3d$, $a+4d$ denote an ascending progression; a denoting the first term and d denoting the common difference.

It may be observed that any term of this series is *equal to the first term plus as many times the common difference as there are preceding terms.*

Suppose the letter l is used to denote any term, and the letter n used to denote the place or position of this term; then the expression for *any general term* will be;

$$l = a + (n-1)\,d.$$

For example the 10th term of the series;

6, 9, 12, 15, 18 would be found

from $l = 6 + (10-1)3 = 6 + 9 \times 3 = 33$.

The letter l is used to denote the desired term, since l is the first letter of the word *last;* the desired term, denoted by l really becomes the *last term* in a given series.

EXAMPLE 83: If the first term of a series is 5, and the common difference is 4, what is the 6th term?

	DATA.
The general equation is;	
$l = a + (n - 1)d$	$a = 5$
Making the proper substitutions gives;	$d = 4$
$l = 5 + (6 - 1)4$	$n = 6$
$= 5 + 20 = 25$. **Answer.**	

PROBLEM 83: If the first term of a series is 8, and the common difference is 5, what is the 10th term?

Tenth term is 53. Answer.

Problem 83a: If the first term of a series is 40, and the common difference is 20, what is the 50th term?

EXAMPLE 84: Find the last term of a decreasing progression if the first term is 60, the number of terms is 20, and the common difference is 3.

For a decreasing arithmetical progression the general equation is;

$$l = a - (n - 1)d$$

	DATA.
Making the proper substitutions in the general equation gives the following:	$a = 60$ $n = 20$ $d = 3$

$$l = 60 - (20 - 1)3$$
$$= 60 - 19 \times 3 = 60 - 57$$

$= 3$. **The last term is therefore 3. Answer.**

PROBLEM 84: The first term of a decreasing series is 100, the number of terms is 40, and the common difference is 2, what is the last term?

The last term is 22. Answer.

Problem 84a: The first term of a decreasing series is 800, the number of terms is 200, and the common difference is 2, what is the last term?

It is sometimes desirable to find the numerical value of the sum of the first few, or of the first n terms of a progression or series.

If the letter S is used to denote the value of the desired sum, the condition may be expressed by;

$$S = \left(\frac{a+l}{2}\right) \times n = \frac{n}{2}(a+l)$$

A simple way to remember this formula is that $S =$ *(the average of the first and last terms)* times *(the number of terms).*

EXAMPLE 85: Find the sum of the first fifty terms of the arithmetical progression, 2, 9, 16, 23 ....

The value of the 50th term is to be found from;

$$l = a + (n+1)d$$
$$= 2 + 61 \times 7$$
$$= 429$$

GIVEN DATA.

$a = 2$

$d = 7$

$n = 50$

COMPUTED DATA.

$l = 429$

The desired sum is found by making the proper substitutions in;

$$S = \frac{n}{2}(a+l)$$
$$= \frac{50}{2}(2+429) = 25 \times 431$$
$$= 10775. \quad \textbf{Answer.}$$

PROBLEM 85: Find the sum of the natural numbers 1, 2, 3, 4, 5 ...., from 1 to 25 inclusive.

$S = \frac{25}{2}(25+1) = 25 \times 13 = 325.$ **Answer.**

Problem 85a: Find the sum of 100 terms of the series 1, 3, 5, 7, 9 ....

EXAMPLE 86: The sum of three numbers in arithmetical progression is 15; the square of the second exceeds the product of the other two by 4. Find the number.

The following expression denotes the first mentioned condition;

$$a + (a+d) + (a+2d) = 15$$

By the second condition; $(a+d)^2 - 4 = a(a+2d)$

$$a^2 + 2ad + d^2 - 4 = a^2 + 2ad$$

Rearranging gives; $d^2 = 4$

$$d = 2$$

The common difference is therefore found to be 2.

Then from the first equation;

$$a + (a + 2) + (a + 4) = 15$$
$$3a + 6 = 15$$
$$3a = 9$$
$$a = 3$$

The numbers are therefore $a = 3$,

$$(a + d) = 3 + 2 = 5,$$
and $(a + 2d) = 3 + 4 = 7$

The square of 5 is $5 \times 5 = 25$
Product of 1st and 3rd is $= 3 \times 7 = 21$

and difference is $= 4$

EXAMPLE 87: Find four integers in arithmetical progression such that their sum shall be 24, and their product 945.

GIVEN DATA.

$S = \frac{n}{2}(a + l)$ and $l = a + (n - 1)d$.

$S = 24$
$n = 4$

From the given conditions, the *sum* of the four integers may be expressed by;

$$a + (a + d) + (a + 2d) + (a + 3d) = 24$$
$$4a + 6d = 24$$
$$a = \frac{24 - 6d}{4} = \frac{12 - 3d}{2}$$

From the first two expressions, by substituting the value of l in the first expression;

$$S = \frac{n}{2}(2a + (n - 1)d)$$
$$24 = 2(2a + 3d). \qquad (n=4)$$
$$24 = 4a + 6d$$
$$a = \frac{24 - 6d}{4} = \frac{12 - 3d}{2}$$

Which verifies the first method above for finding an expression for a.

According to the second condition given in the example;

$$a \times (a + d) \times (a + 2d) \times (a + 3d) = 945$$

Substituting in the last equation the value $d = \frac{12 - 2a}{3}$ obtained from one of the previous equations gives;

$$a\left(a + \frac{12 - 2a}{3}\right)[a + \tfrac{2}{3}(12 - 2a)]\ (a + 12 - 2a) = 945$$

Which may be reduced to the following;

$$a(12+a)\ (24-a)\ (12-a)=8505$$

If in this expression the values 1, 2 and 3 are successively substituted for a, it will be found that 3 will fulfill the condition, so that $a=3$. That is, the first number of the arithmetical progression is 3.

Substituting the proper values in;

$$S=\frac{n}{2}(a+l) \text{ gives } 24=\tfrac{4}{2}(3+l)$$

From which l is found as follows;

$$24=2(3+l)=6+2l$$

$$l=\tfrac{18}{2}=9 \qquad \text{(This serves as a check on the result)}$$

The last of the four numbers is therefore 9.

Substituting the value of $a=3$ in;

$$a+(a+d)+(a+2d)+(a+3d)=24$$

gives the common difference $d=2$.

If $a=3$, then the second number is $3+2=5$; the third number is $5+2=7$, and the fourth number is $7+2=9$.

EXAMPLE 88: Find the sum of all positive integers consisting of three digits which are multiples of 11.

The last term of the progression cannot be greater than 999; the nearest multiple of 11 is 990. The last term is therefore 990.

The first term is 110.
The difference is 11.

$$l=a+(n-1)d=a+nd-d$$

from which $n=\dfrac{l-a+d}{d}$

$$=\frac{990-110+11}{11}=\frac{891}{11}=81$$

$$S=\frac{n}{2}(a+l)$$

$S=\frac{81}{2}(110+990)=81\,\frac{1100}{2}=81\times 550=44550.$ **Answer.**

LESSON XXIII

GEOMETRICAL PROGRESSION

A geometrical progression is a series of numbers so arranged that the *quotient* of any term by the preceding term is always the same.

The letters **G. P.** are sometimes used to designate geometrical progression.

The constant quotient mentioned above is also called the **ratio,** and is designated by r.

A *geometrical progression* is a **progression by quotients.**

The following is an example of a *decreasing* geometrical progression;

$$64 : 32 : 16 : 8 : 4 : 2 : 1$$

while the following illustrates an *increasing* geometrical progression

$$1 : 3 : 9 : 27 : 81 : 243 \ \ldots.$$

From the last example it is evident that $\frac{3}{1} = 3$, or the ratio $r = 3$. If the first term of any geometrical progression be denoted by a, then the second term may be denoted by $a\,r$, the third term by $a\,r^2$, the fourth by $a\,r^3$, etc.

Any term that has $n - 1$ terms preceding it may be expressed by;

$$l = a\,r^{(n-1)}$$

The sum of any number, (as n) of terms of a geometrical progression may be designated by;

$S = \dfrac{l\,r - a}{r - 1}$ in which a denotes the first term and l denotes the nth term.

EXAMPLE 89: Find the 5th term of the progression;

2, 4, 8, 16 ....

DATA.

$a = 2$

$r = 2$

$n = 5$

Substituting the proper values in the equation;

$$l = a\,r^{n-1}$$

gives $l = 2 \times 2^{5-1} = 2 \times 2^4 = 32$. **Answer.**

PROBLEM 89: Find the 8th term of the progression;

2, 6, 18, 54.

8th term $= 2 \times 3^7 = 2 \times 2187 = 4374$. **Answer.**

Problem 89a: Find the 6th term of the progression;

4, 12, 36, 108.

EXAMPLE 90: Find the sum of eight terms of the progression;

2, 6, 18, 54, 162.

DATA.

$a = 2$

$n = 8$

$r = 3$

First find the value of the 8th term.

$l = 2 \times 3^7 = 4374$

Then the sum of the first eight terms is found from;

$$S = \frac{lr - a}{r - 1} = \frac{4374 \times 3 - 2}{3 - 1} = 6560. \quad \textbf{Answer.}$$

PROBLEM 90: Find the sum of ten terms of the progression;

2, 6, 18, 54, 162, ...

Sum of ten terms = 59048. Answer.

Problem 90a: Find the sum of five terms of the progression;

2, 4, 8, 16, 32.

The application of geometrical progression may be appreciated by the following consideration;

Suppose it is desired to find the absolute value of the recurring or repeating decimal .333333.....

This decimal may be written in form,

$$\frac{33}{100} + \frac{33}{(100)^2} + \frac{33}{(100)^3} + \ldots..$$

In this case $a = \frac{33}{100}$, $r = \frac{1}{100}$ and the value to infinity of the series is;

$$S_\infty = \frac{a}{1 - r} = \frac{\frac{33}{100}}{1 - \frac{1}{100}} = \frac{.33}{.99} = \frac{1}{3}$$

To find the absolute value of the recurring decimal .5555.....

$$S_\infty = \frac{a}{1 - r} = \frac{.55}{.99} = \frac{5}{9}$$

To find the absolute value of the recurring decimal .66666.....

$$S_\infty = \frac{a}{1 - r} = \frac{.66}{.99} = \frac{6}{9} = \frac{2}{3}$$

See Lesson VI, page 26.

EXAMPLE 91: A man purchased 10 bushels of wheat on the condition that he should pay 1 cent for the first bushel, 3 cents for the second, 9 for the third, and so on to the last. What did he pay for the last bushel, and for all ten bushels?

The last or the tenth bushel cost;

$$l = 1 \times 3^9 = 19683 \text{ cents or } \$196.83.$$

The ten bushels cost;

$$S = \frac{19683 \times 3 - 1}{3 - 1} = 29524 \text{ cents or } \$295.24.$$

PROBLEM 91: What debt can be paid in 12 months, by paying $1 the first month, $2 the second month, $4 the third month, and so on, each payment being double the preceding one? What will be the last payment?

Debt is $4095, and last payment is $2048. Answer.

Problem 91a: A farmer planted 4 bushels of corn, and the harvest yielded 32 bushels; these were planted the next year, and yielded 256 bushels; these were planted the following year and again yielded 8 fold; this process was continued during 16 years. How many bushels were the result of the last harvest, and what was the total number of bushels of corn handled during the 16 years?

Result of last harvest = 140737488355328 bushels.

LESSON XXIV

CLOCK PROBLEMS

As the many questions that may be asked regarding the various time relations of the hands of a clock offers a very attractive field for examiners, a few of the numerous and impractical problems, which offer great possibilities for mental training will be considered.

The *fundamental relation* of the various positions of the hands of a clock or of a watch *are of considerable importance* and should be carefully considered.

For instance at 12 o'clock the two hands *coincide*. This time and position of the two hands *might* be taken as the datum from which all other relations could be computed; however other positions may offer easier solutions for some problems.

It may be advisable to consider a circle and circular measure, in connection with clock problems.

A circle may be defined as a curve, every point of which is equally distant from a point within the curve, called its center. The term circle is assumed to mean the area included by the curved line; while the curved line itself is referred to as the circumference of the circle. Portions of the circumference are referred to as *arcs*. Every circumference or circle is supposed to be made up of 360 units called *degrees* (expressed by $^\circ$; thus 45° means forty-five degrees).

A semicircle $= \frac{360}{2} = 180^\circ$; one fourth of a circumference $= \frac{360}{4} = 90^\circ$, and is called a *quadrant*. One-twelfth of a circumference will equal 30°.

The minute hand of a watch passes over 360° while the hour hand passes over 30°. The minute hand moves 12 times as fast as the hour hand.

EXAMPLE 92: When will the hands of a watch next coincide, after 12 o'clock?

When the hour hand has passed over an angle of x degrees, the minute hand has passed over an angle of $x + 360^\circ$.

Since the minute hand moves 12 times as fast as the hour hand, the following relation must be true;

$$12x = x + 360^\circ$$
$$11x = 360$$
$$x = \tfrac{360}{11} \text{ degrees.}$$

Since the hour hand moves over 30° in one hour; or during 60 minutes; one degree must correspond with two minutes, for this hand.

The actual time, in minutes, corresponding with $\frac{360}{11}$ degrees, must be

$$\tfrac{360}{11} \times 2 = 65\tfrac{5}{11} \text{ minutes.}$$

The two hands next coincide at **$65\frac{5}{11}$minutes after 12 o'clock; or at $1\text{–}5\frac{5}{11}$ o'clock.** That is at $5\frac{5}{11}$ minutes past one o'clock.

Answer.

The next coincidence will be $1\text{–}5\frac{5}{11} + 1\text{–}5\frac{5}{11}$ or at $2\text{–}10\frac{10}{11}$, and so on.

PROBLEM 92: Where is the hour hand of a watch, when the minute hand is at 12–15 o'clock?

At $1\frac{1}{4}$ minutes past 12. Answer.

Problem 92a: Starting at 12 o'clock what is the value of the angles, in degrees, each of the two hands have passed over at the end of 25 minutes?

EXAMPLE 93: At what time between one and two o'clock do the hands of a watch coincide?

The angle between the hands, at one o'clock is 30°. After one o'clock the hands begin to move and the minute hand moves 12 times as rapidly as the hour hand. In one hour the hour hand passes over 30° of angle.

Let x denote the speed of the hour hand; then $12x$ denotes the speed of the minute hand.

The condition stated in the example may therefore be expressed by;

$$12x - 30° = x$$

From which $$11x = 30°$$

and $$x = 2\tfrac{8}{11}° = \tfrac{30}{11}°$$

For the minute hand a movement of 1 degree $= \frac{60}{360} = \frac{1}{6}$ of a minute. and for the hour hand 1 degree $= 12 \times \frac{1}{6} = 2$ minutes. During one complete rotation of either hand it sweeps over 360 degrees of angle or of arc.

If $x = \frac{30}{11}$degrees and one degree for the hour hand means 2 minutes, in the given case the time interval must be $\frac{30}{11} \times 2 = \frac{60}{11} = 5\frac{5}{11}$ minutes.

At $5\frac{5}{11}$ minutes past one o'clock the two hands coincide.

Answer.

PROBLEM 93: At what time between 2 and 3 o'clock are the hands of a clock together?

$$12x - 60° = x$$
$$11x = 60°$$
$x = \frac{60}{11}°$ **At $10\frac{10}{11}$ minutes after 2 o'clock. Answer.**

The minute hand passes over $\frac{360}{60} = 6°$ every minute.
The hour hand passes over $\frac{360}{12}° = 30°$ every hour or $\frac{1}{2}°$ every minute; or 1 degree every 2 minutes.

The *converse* of the last problem would be; **when the hands of a clock coincide between 2 and 3 o'clock, what time is it?**

EXAMPLE 94: What time is it when the hands of a watch coincide, between 3 and 4 o'clock?

At 3 o'clock the angle between the hands is 90°. The condition for the coincidence of the two hands after this time is; $12x - 90° = x$ where x denotes the speed of the hour hand.

Then $11x = 90°$ or $x = \frac{90}{11}°$
The hour hand has moved over $\frac{90}{11}° = 8\frac{2}{11}$ degrees, and since the minute hand moves 12 times as rapidly, it must have moved $12 \times 8\frac{2}{11}° = 98\frac{2}{11}°$.
Since for the minute hand $6° = 1$ minute, the minute hand must indicate $\frac{1080}{66} = 16\frac{4}{11}$ minutes after three o'clock.
For the hour hand $\frac{1}{2}° = 1$ minute. Therefore if this hand has passed over $8\frac{2}{11}°$, it also must indicate $\frac{180}{11} = 16\frac{4}{11}$ minutes after 3 o'clock.

EXAMPLE 95: At what time between 2 and 3 o'clock are the hands of a watch at right angles with each other?

The condition is expressed by;

$$12x - 60° = x + 90°$$

From which $11x = 150°$

and $x = \frac{150}{11}° = 13\frac{7}{11}°$

$13\frac{7}{11} \times 2 = 27\frac{3}{11}$ **minutes past 2 o'clock. Answer.**

PROBLEM 95: At what time between 1 and 2 o'clock are the hands of a watch at right angles with each other?

At $21\frac{9}{11}$ minutes past 2 o'clock. Answer.

Problem 95a: At what times between 3 and 4 o'clock are the hands of a watch at right angles with each other?

A rough check on such problems as the preceding, should always be applied by the student, in some such manner as follows; From the conditions mentioned in Problem 95a, it is evident that the hour hand must be located below the figure 3 and if the minute hand is 90° in *advance* of the position of the hour hand it must be beyond the figure 6, since there are only 90° between the figure 3 and the figure 6. This means that it must be more than 3-30. Another position will be when the minute hand is between 12 and 1. A comprehensive problem of no great practical importance would be asking **at what times during the day are the hands of a clock at right angles?** Obvious times are 3 o'clock and 9 o'clock.

EXAMPLE 96: What is the time interval, in minutes, between the two perpendicular positions of the hands of a watch between the hours of 4 and 5 o'clock?

At some time between 4 o'clock and 4–10 the two hands are perpendicular with each other. As the two hands move around they will later coincide, and still later will again be perpendicular with each other. The time interval is required between the two perpendicular positions.

Let x denote the angle passed over by the *hour hand.*

Then $12x$ will express the angle passed over in the same time by the minute hand.

At 4 o'clock the angle between the two hands is 120°.

After the hour hand has moved x degrees, the minute hand has moved $12x$ degrees.

Starting with 12 o'clock* as datum, the following relation is true for the first perpendicular position;

$$\frac{\text{Angle passed over by minute hand}}{\text{Angle passed over by hour hand}} = \frac{x + 4(360°)}{x} = \tfrac{12}{1};$$

from which $\quad 12x = x + 1440°$

$$x = \tfrac{1440}{11}°$$

For the second perpendicular position the following is true;

$$\frac{x^1 + 4(360°) + 180°}{x^1} = \tfrac{12}{1}$$

From which $x^1 = \dfrac{1620}{11}$

The difference between the two positions is $\tfrac{1620}{11} - \tfrac{1440}{11} = \tfrac{180}{11}$ degrees

The time interval is therefore $2 \times \tfrac{180}{11} = \tfrac{360}{11} = 32\tfrac{8}{11}$ **minutes.**

Answer.

* Four o'clock could be assumed as datum if desired.

TABLE OF MEASURES

Dry Measure

2 pints............1 quart—qt.	4 pecks............1 bushel—bu.
8 quarts............1 peck—pk.	36 bushels............1 chaldron

Liquid or Wine Measure

4 gills............1 pint—pt.	U. S. Standard Gallon 231 cubic inches
2 pints............1 quart—qt.	Beer gallon.....282 cubic inches
4 quarts............1 gallon—gal.	36 beer gallons............1 barrel
31½ gallon........1 barrel—bbl.	
2 barrels.......1 hogshead—hhd.	

Time Measure

60 seconds............1 minute	24 hours............1 day
60 minutes............1 hour	7 days............1 week

4 weeks............1 lunar month
28, 29, 30, or 31 days............1 calendar month
30 days (in computing interest)............1 month
52 weeks and 1 day or 12 calendar months............1 year
365 days, 5 hours, 48 minutes, and 49 seconds............1 solar year

Circular Measure

60 seconds............1 minute	90 degrees............1 quadrant
60 minutes............1 degree	4 quadrants or 360 degrees 1 circle
30 degrees............1 sign	

Long Measure

12 inches............1 foot	40 rods............1 furlong
3 feet............1 yard	8 furlongs........1 statute mile
5½ yards............1 rod	3 miles............1 league

Square Measure

144 square inches..1 square foot	40 square rods............1 rood
9 square feet......1 square yard	4 roods............1 acre
30¼ square yards..1 square rod	640 acres..........1 square mile

Cubic Measure

1728 cubic inches...1 cubic foot	128 cubic feet ...1 cord (wood)
27 cubic feet........1 cubic yard	40 cubic feet...1 ton (shipping)

2,150.42 cubic inches............1 standard bushel
268.8 cubic inches............1 standard gallon
1 cubic foot............about four-fifths of a bushel

Surveyors' Measure

7.92 inches............1 link	4 rods............1 chain
25 links............1 rod	

10 square chains or 160 square rods............1 acre
640 acres............1 square mile
36 square miles (6 miles square)............1 township

Cloth Measure

2¼ inches	1 nail	4 quarters	1 yard
4 nails	1 quarter		

Mariners' Measure

6 feet	1 fathom	5,280 feet	1 statute mile
120 fathoms	1 cable length	6,085 feet	1 nautical mile
7½ cable lengths	1 mile		

Miscellaneous

3 inches	1 palm	18 inches	1 cubit
4 inches	1 hand	21.8 inches	1 Bible cubit
6 inches	1 span	2½ feet	1 military pace

TABLE OF WEIGHTS

Troy Weight

24 grains (gr)	1 pennyweight—pwt.
20 pennyweights	1 ounce—oz.
3.2 grains	1 carat, diamond weight

By this weight gold, silver and jewels only are weighed. The ounce and pound in this are the same as in apothecaries' weight.

Apothecaries' Weight

20 grains	1 scruple	8 drachms	1 ounce
3 scruples	1 drachm	12 ounces	1 pound

Avoirdupois Weight

16 drachms	1 ounce	4 quarters	100 weight—cwt.
16 ounces	1 pound	20 hundred weight	1 ton
25 pounds	1 quarter—qd.		

5,760 grains apothecaries' or troy weight	1 pound
7,000 grains avoirdupois weight	1 pound

Therefore, 144 pounds avoirdupois equal 175 pounds apothecaries' or troy.

Liquids

1 gallon oil weighs	7.32 pounds avoirdupois
1 gallon distilled water	8.33 pounds
I gallon sea water	8.55 pounds
1 gallon proof spirits	7.68 pounds

1 grain = 0.00229 ounce.
= 0.064799 gram.
= 0.00014 pound.
1 ounce = 437.5 grains.
= 28.3495 grams.
= 0.0625 pound.
1 pound = 7,000 grains.
= 453.6 grams.
= 0.4536 kilogram.

1 Long Ton = 2,240 pounds.
1 Short Ton = 2,000 pounds.
1 Metric Ton = 1,000 kilograms.
1 Acre = 43,560 square feet.
= 4840 square yards.
= .0015625 square mile.
1 Square Mile = 27,878,400 square feet
= 3,097,600 square yards.
1 Square Yard = 1296 square inches.

METRIC EQUIVALENTS IN LINEAR MEASURE

Linear Measure

1 millimeter.........0.03937 inch	1 inch...........2.54 centimeters
1 centimeter...........0.3937 in.	1 foot...........3.048 decimeters
1 decimeter..3.937 in. = 0.328 ft.	1 yard..............0.9144 meter
1 meter....39.37 in. = 1.0936 yds.	1 rod..........0.5029 dekameter
1 dekameter.........1.9884 rods	
1 hectometer..........328.091 ft.	1 mile..........1.6093 kilometers
1 kilometer.........0.62137 mile	
1 kilometer.....1093.63890 yards	
1 myriameter...10936.3890 yards	

Square Measure

1 sq. centimeter....0.1550 sq. in.	1 sq. inch...6.452 sq. centimeters
1 sq. decimeter......0.1076 sq. ft.	1 sq. foot..9.2903 sq. decimeters
1 sq. meter.........1.196 sq. yd.	1 sq. yd........0.8361 sq. meters
1 are................3.954 sq. rd.	1 sq. rod.............0.2529 are.
1 hektar..............2.47 acres	1 acre.............0.4047 hektar
1 sq. kilometer.......0.386 sq. m.	1 sq. m.......2.59 sq. kilometers

Measure of Volume

1 cu. centimeter.....0.061 cu. in.	1 cu. in....16.39 cu. centimeters
1 cu. decimeter.....0.0353 cu. ft.	1 cu. ft....28.317 cu. decimeters
1 cu. meter.........1.308 cu. yd.	1 cu. yd.........0.7646 cu. meters
1 stere...............0.2759 cord	1 cord................3.624 steres
1 liter......... { 0.908 qt. dry	1 qt. dry.............1.101 liters
{ 1.0567 qt. liq.	1 qt. liq..............0.9463 liter
1 dekaliter.... { 2.6417 gal.	1 gal............0.3785 dekaliter
{ .135 pks.	1 peck............0.881 dekaliter
1 hektoliter........2.8375 bushels	1 bushel.........0.3524 hektoliter

Weights

1 milligram........0.0154 grains	1 kilogram............2.2046 lbs.
1 centigram........0.1543 grains	1 metric ton..1.1023 English ton
1 decigram.........1.5434 grains	1 ounce.............28.85 grams
1 gram............15.4340 grains	1 lb..............0.4536 kilogram
1 gram............0.03527 ounce	1 English ton..0.9072 metric ton
1 decagram......154.3402 grains	
1 hectogram 3.527 oz. Avoirdupois	

APPROXIMATE METRIC EQUIVALENTS

1 decimeter............4 inches	1 liter 1.06 qt. liq. = 0.9 qt. dry
1 meter................1.1 yards	1 hektoliter..........2⅝ bushels
1 kilometer...........⅝ of mile	1 kilogram.........2 1/5 pounds
1 hektar...............2½ acres	1 metric ton........2,200 pounds
1 stere or cu. meter ¼ of a cord	

1 quart............0.94636 liter

1 gallon..........3.78543 liters

RULES FOR COMPUTING INTEREST

The following will be found to be excellent rules for finding the interest on any principal for any number of days. When the principal contains cents, point off four places from the right of the result to express the interest in dollars and cents. When the principal contains dollars only, point off two places.

Two per Cent.—Multiply the principal by the number of days to run, and divide by 180.

Two and one-half per Cent.—Multiply by number of days, and divide by 144.

Three per Cent.—Multiply by number of days, and divide by 120.

Three and one-half per Cent.—Multiply by number of days, and divide by 102.86.

Four per Cent.—Multiply by number of days, and divide by 90.

Five per Cent.—Multiply by number of days, and divide by 72.

Six per Cent.—Multiply by number of days, and divide by 60.

Seven per Cent.—Multiply by number of days, and divide by 51.43.

Eight per Cent.—Multiply by number of days, and divide by 45.

Nine per Cent.—Multiply by number of days, and divide by 40.

Ten per Cent.—Multiply by number of days, and divide by 36.

Twelve per Cent.—Multiply by number of days, and divide by 30.

Fifteen per Cent.—Multiply by number of days, and divide by 24.

The 4th root is the square root of the square root.

The 6th root is the square root of the cube root.

The 9th root is the cube root of the cube root.

EXAMINATION IN ALGEBRA

1. Express the following, using positive exponents;

$$x^3 y^{-5}; \qquad 7x^{\frac{1}{3}} y^{-\frac{1}{3}}; \qquad m^{-2} n^2 p^{-\frac{2}{3}}$$

2. Express the following, using radical signs;

$$a^{\frac{2}{3}}; \quad 7x^{\frac{1}{3}} y^{-\frac{2}{3}}; \quad x^{\frac{1}{2}} y^{\frac{1}{6}}; \quad a^{\frac{n}{3}}; \quad 5x^{\frac{1}{4}} y^{\frac{4}{5}}$$

3. Write with fractional exponents the following;

$$\sqrt{b^3}; \qquad \sqrt[3]{x^5}; \qquad \sqrt[5]{\sqrt[3]{a^2}}$$

4. Perform the operations indicated, and simplify;

$$a^{\frac{1}{2}} a^{\frac{1}{3}}; \quad \left(\tfrac{4}{5}\right)^{\frac{1}{2}} \times \left(3\tfrac{1}{3}\right)^{\frac{1}{2}}; \quad \frac{a^{\frac{1}{2}} x}{\sqrt{a}}$$

5. Simplify; $\dfrac{\frac{a}{2} - \frac{b}{3}}{\frac{a}{2} + \frac{b}{3}}$; $\dfrac{2x + \frac{x}{x-2}}{2x - \frac{x}{x-2}}$

6. Separate into factors;

$$3a^2 - 6ab + 9\,a^2 b^2; \qquad a^4 + a^2 + 1$$

7. Find three numbers whose sum is 20, and the first plus twice the second plus three times the third is 44, and twice the sum of the first two minus four times the third is -14.

8. Two passengers together have 500 pounds of baggage. One pays \$1.25 and the other \$1.75 for excess weight. If the baggage had belonged to one person he would have paid \$4.00 for excess. How much baggage is allowed free to each person.

9. Solve;

$$\begin{aligned} x + y &= 11 \qquad (1)\\ y + z &= 13 \qquad (2)\\ z + x &= 12 \qquad (3) \end{aligned}$$

10. Solve;

$$\begin{aligned} 2x + 4y - 3z &= 3 \qquad (1)\\ 3x - 8y + 6z &= 1 \qquad (2)\\ 8x - 2y - 9z &= 4 \qquad (3) \end{aligned}$$

Solution of the Foregoing Examination

1. $\dfrac{x^3}{y^5}$; $\quad 7\,\dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}$; $\quad \dfrac{n^2}{m^2 p^{\frac{2}{3}}}$

2. $\sqrt[3]{a^2}$; $\quad 7\dfrac{x^{\frac{1}{3}}}{y^{\frac{2}{3}}} = 7\,\dfrac{\sqrt[3]{x}}{\sqrt[3]{y^2}}$; $\quad \sqrt{x}\sqrt[6]{y}$; $\quad \sqrt[3]{a^n}$; $\quad 5\sqrt[4]{x}\quad \dfrac{1}{\sqrt[5]{y^4}}$

3. $b^{\frac{3}{2}}$; $\quad x^{\frac{5}{3}}$; $\quad \sqrt[5]{a^{\frac{2}{3}}} = \left(a^{\frac{2}{3}}\right)^{\frac{1}{5}} = a^{\frac{2}{15}}$

4. $a^{\frac{5}{6}}$ (adding exponents); $\dfrac{\sqrt{4}}{\sqrt{5}} \times \dfrac{\sqrt{10}}{\sqrt{3}} = \dfrac{2\sqrt{10}}{\sqrt{15}} = \dfrac{2\sqrt{15}\sqrt{10}}{15}$

$$= \frac{2 \times \sqrt{5} \times \sqrt{3} \times \sqrt{2} \times \sqrt{5}}{15} = \frac{2 \times 5 \times \sqrt{2} \times \sqrt{3}}{15} = \tfrac{2}{3}\sqrt{6};$$

$$\frac{x\sqrt{a}}{\sqrt{a}} = x.$$

5. $\dfrac{3a - 2b}{3 + 2b}$; $\quad \dfrac{2x^2 - 3x}{2x^2 - 5x}$

6. $3a(a - 2b + 3ab^2)$.

7. Let x, y, and z denote the three numbers, then, according to the first mentioned condition,

$$x + y + z = 20 \qquad (1)$$

By second condition; $\quad x + 2y + 3z = 44 \qquad (2)$

By third condition; $\quad 2(x + y) - 4z = -14$

or $\quad 2x + 2y - 4z = -14 \qquad (3)$

The three equations may be designated as (1), (2), and (3).

Multiplying the first equation by 2 and subtracting the result from (3) gives;

$$\begin{array}{r} 2x + 2y - 4z = -14 \\ 2x + 2y + 2z = 40 \\ \hline -6z = -54 \end{array}$$

From which $\quad z = 9$

Substituting this value for z in (2) and (3) gives;

$$x + 2y + 27 = 44 = x + 2y = 17$$
$$2x + 2y - 36 = -14 = 2x + 2y = 22$$

Subtracting the last two equations gives $x = 5$.

Substituting in (1) the values $x = 5$ and $z = 9$, gives;

$$5 + y + 9 = 20$$
$$y = 20 - 14 = 6$$

Verification;

$$\begin{aligned} 5+6+9 &= 20 \\ 5+2\times 6+3\times 9 &= 44 \\ 2(5+6)-4\times 9 &= -14 \end{aligned}$$

Another method is as follows;

$$\begin{aligned} x+y+z &= 20 \\ x+2y+3z &= 44 \\ 2x+2y-4z &= -14 \end{aligned}$$

Subtracting the first two equations gives; $y+2z = 24$
Subtracting the last two equations gives; $2y+10z=102$ after multiplying the second equation by 2.

These two resulting equations may be arranged as follows and subtracted.

$$\begin{aligned} 2y+4z &= 48 \\ -2y-10z &= -102 \\ \hline -6z &= -54 \end{aligned}$$

from which $z=9$.

Substituting this value for z in first two of the original equations gives;

$$\begin{aligned} x+y+9 &= 20 \\ x+2y+27 &= 44 \end{aligned} \quad \text{or} \quad \begin{aligned} x+y &= 11 \\ x+2y &= 17 \end{aligned}$$

Subtracting gives; $-y=-6$ or $y=6$.

Substituting the values found for y and z in the first of the original equations gives;

$$x+6+9=20 \quad \text{or} \quad x=5.$$

The three numbers desired are therefore, 5, 6, and 9.

8. Let x denote the pounds of baggage allowed free to each person.

Let y denote the pounds of baggage belonging to one passenger; then $500-y$ will denote the pounds of baggage belonging to the other passenger.

Then $\dfrac{125}{y-x}$ = cost per single pound of excess baggage.

and $\dfrac{175}{500-y-x}$ = cost per single pound of excess baggage.

Then $\dfrac{125}{y-x}=\dfrac{175}{500-y-x}$ (a)

Also $\dfrac{400}{500-x}$ = cost per single pound of excess baggage.

Then $\dfrac{400}{500-x}=\dfrac{125}{y-x}$ (b)

(a) reduces to; $50x - 300y = -62500$

or to $x - 6y = -1250$

Multiplying by 11 gives $-66y + 11x = -13750$

(b) reduces to $400y - 275x = 62500$

or to $16y - 11x = 2500$

$$\begin{array}{r} -66y + 11x = -13750 \\ \hline \end{array}$$

Adding gives; $-50y = -11250$

$y = 225$ pounds.

If one passenger has 225 pounds, the other must have $500 - 225 = 275$ pounds.

Also $x = 100$ pounds; amount allowed free to each person.

Charge is one cent per pound of excess baggage.

9. Subtracting equation (2) from (1) gives;

$$x - z = -2$$

Subtracting this from equation (3) gives;

$$-2z = -14 \quad \text{or} \quad z = 7.$$

Substituting this value in (2) gives $y = 6$. Substituting this value in (1) gives $x = 5$.

10. Multiplying equation (1) by 2 gives;

$4x + 8y - 6z = 6$ Add equation (2) to last equation; $3x - 8y + 6z = 1$

Adding gives; $7x = 7$ or $x = 1$

Multiply equation (3) by 4 giving;

$$32x - 8y - 36z = 16$$

Substitute value 1 for x, giving;

$$-8y - 36z = -16$$

Substitute value 1 for x in equation (2) giving,

$$-8y + 6z = -2$$

Subtracting last two equations gives;

$$\begin{array}{r} -8y - 36z = -16 \\ -8y + 6z = -2 \\ \hline -42z = -14 \end{array}$$

$$z = \tfrac{14}{42} = \tfrac{1}{3}$$

Multiplying equation (2) by 3 and substituting $x = 1$ gives;

$-24y + 18z = -6$. Mult. (3) by 2 and substituting $x = 1$ gives; $-4y - 18z = -8$

Adding gives; $-28y = -14$

$$y = \tfrac{1}{2}$$

Substituting $x = 1$ and $y = \frac{1}{2}$ in equation (1) gives; $z = \frac{1}{3}$.

EXAMINATION FOR ADMISSION, BROWN UNIVERSITY

ALGEBRA, Wednesday, September 20, 1912

1. (a) Divide $x^{\frac{6}{5}}+a^{\frac{9}{2}}$ by $x^{\frac{2}{5}}+a^{\frac{3}{2}}$

   (b) Determine the least common multiple of

   $$x^2-a^2; \qquad x+a; \qquad x^3-a^3$$

2. Find the square root of 347, correct to two decimal places.

3. Solve for x, y and z,

$$3x-4y+2z=-3$$
$$2x+y-3z=8\tfrac{1}{2}$$
$$x+2z=-3$$

4. Solve for x; $\qquad 3ax^2+2bx+c=0.$

5. A man buys 8 lbs. of tea and 5 lbs. of sugar for \$2.39; and at another time 5 lbs. of tea and 8 lbs. of sugar for \$1.64, the price being the same as before. What were the prices?

6. A man had one dollar in silver and copper coins; each copper coin was worth as many cents as there were silver coins; there were in all 27 coins. How many of each were there?

7. The sum of three numbers in arithmetical progression is 15; the square of the second exceeds the product of the other two by 4. Find the numbers.

Answers to the foregoing examination;

1. (a)

$$\begin{array}{l|l}
 & x^{\frac{4}{5}}-a^{\frac{3}{2}}x^{\frac{2}{5}}+a^{\frac{6}{2}} \quad \text{Answer} \\
\cline{2-2}
x^{\frac{2}{5}}+a^{\frac{3}{2}} & x^{\frac{6}{5}}+a^{\frac{9}{2}} \\
 & x^{\frac{4}{5}}+a^{\frac{3}{2}}x^{\frac{4}{5}} \\
 & \quad -a^{\frac{3}{2}}x^{\frac{4}{5}}+a^{\frac{9}{2}} \\
 & \quad -a^{\frac{3}{2}}x^{\frac{4}{5}}-a^{\frac{6}{2}}x^{\frac{2}{5}} \\
 & \qquad a^{\frac{6}{2}}x^{\frac{2}{5}}+a^{\frac{9}{2}} \\
 & \qquad a^{\frac{6}{2}}x^{\frac{2}{5}}+a^{\frac{9}{2}}
\end{array}$$

(b) Factoring the expression gives;

$$(x+a)\ (x-a), \quad (x+a), \quad (x-a)\ (x^2+ax+a^2)$$

The least common multiple is therefore;

$(x+a)\ (x-a)\ (x^2+ax+a^2) = x^4-ax^3+2a^2x^2-a^3x+a^4.$

Answer.

2. 18.260. **Answer.**

3. $x = 1$; $y = \frac{1}{2}$, and $z = -2$. **Answer.**

4. Dividing each term by $3a$ gives;

$$x^2 + \frac{2}{3}\frac{b}{a}x + \left(\frac{1}{3}\frac{b}{a}\right)^2 = \frac{b^2 - 3ac}{9a^2}$$

Extracting the square root of each member of the equation gives;

$$x + \frac{b}{3a} = \pm\sqrt{\frac{b^2 - 3ac}{9a^2}} = \pm\frac{\sqrt{b^2 - 3ac}}{3a}$$

$$x = -\frac{b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{3a}$$

$$= \frac{1}{3a}\left(-b \pm \sqrt{b^2 - 3ac}\right)$$

5. Let x denote the price of tea.

Let y denote the price of sugar.

Then by condition; $8x + 5y = 239$ (1)

Also; $5x + 8y = 164$ (2)

Multiplying (1) by 8, and (2) by 5 gives;

$$64x + 40y = 1912$$
$$25x + 40y = 820$$

Subtracting gives; $39x = 1092$

And $x = 28$

Tea cost 28 cents per pound. Answer.

Substituting value $x = 28$ in equation (2) gives;

$$140 + 8y = 164$$

From which $y = 3$

Sugar cost 3 cents per pound. Answer.

Verification; 8 lbs. × 28 cts. + 5 lbs. × 3 cts. = 239 cents = \$2.39.
5 lbs. × 28 cts. + 8 lbs. × 3 cts. = 164 cents = \$1.64.

6. Let x denote the number of silver coins.

Let y denote the number of copper coins.

Each copper coin is worth x cents.

Each silver coin is worth y cents.

It is evident that the number of copper coins times the value in cents of each coin, added to the product of the number of silver coins times the value in cents of each silver coin must total 100 cents.

All the silver coins are worth xy cents and all the copper coins are worth yx cents.

Then by the conditions;

$$yx + yx = 100 \text{ or } 2xy = 100$$

From which $y = \dfrac{50}{x}$

Also by condition; $x + y = 27$.

Substituting the value $y = \dfrac{50}{x}$ in the last equation gives;

$$x + \frac{50}{x} = 27$$

$$x^2 + 50 = 27x$$

$$x^2 - 27x = -50$$

$$x^2 - 27x + (\tfrac{27}{2})^2 = (\tfrac{27}{2})^2 - 50 = \tfrac{729}{4} - 50 = \tfrac{729-200}{4} = \tfrac{529}{4}$$

Extracting square root; $x - \frac{27}{2} = \pm\sqrt{\frac{529}{4}} = \pm\frac{23}{2}$

$$x = \tfrac{27}{2} \pm \tfrac{23}{2}$$

$$= \tfrac{50}{2} \text{ or } \tfrac{4}{2} = 25 \text{ or } 2$$

Substituting the value $x = 25$ in $x + y = 27$, gives $y = 2$. Substituting the value $x = 2$ then $y = 25$.

If there are 25 silver coins there are 2 copper coins. The silver coins being worth 2 cents each, and the copper coins 25 cents each.

$$2 \times 25 + 25 \times 2 = 100$$

If there are 2 silver coins then there are 25 copper coins.

Then $\quad 25 \times 2 = 2 \times 25 = 100$

7. By first condition;

$x + (x + d) + (x + 2d) = 15$. (See page 133 on progression.)

By second condition;

$$(x + d)^2 - 4 = x(x + 2d)$$

$$x^2 + 2dx + d^2 - 4 = x^2 + 2dx$$

From which $\quad d^2 = 4$

Or $\quad d = \pm 2$

Substituting the value $d = 2$ in the first equation gives;

$$x + x + 2 + x + 4 = 15 \text{ or } 3x = 9 \qquad x = 3$$

The first number is therefore 3, the second is $3 + 2 = 5$ and the third is $3 + 4 = 7$.

If the value $d = -2$ is used then from

$$x + (x + d) + (x + 2d) = 15$$

$x = 7$. Then the second number is $7 - 2 = 5$, and the third is $7 - 4 = 3$.

Verification; $\quad 3 + 5 + 7 = 15$

$\quad 25 - 4 = 21$

Second case; $\quad 7 + 5 + 3 = 15$

$\quad 25 - 4 = 21$

DARTMOUTH COLLEGE

ENTRANCE EXAMINATION—SEPTEMBER, 1915

(Students taking examination in Elementary Algebra complete ($1\frac{1}{2}$ units credit) will take questions 2 and 4 of mathematics A1 and all the questions of mathematics A2. Such students will be allowed 2 hours and 20 minutes.)

Mathematics A1 (Algebra to Quadratics)

1. Solve each of the following equations for x:

(*a*) $\dfrac{2x+7}{4}-\dfrac{3x+8}{5x+3}=\dfrac{4x+3}{8}$

(*b*) $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-a-b}$

2. (*a*) Factor:

$75xy^3 - 130x^2y^4 - 9x^3y^5$

$x^3y - xy^3 + x^2y - xy^2$

$4(x-3)+x(x-3)(4x-4)$

(*b*) Simplify:

$$\frac{x}{1+\frac{x}{y}}+\frac{y}{1+\frac{y}{x}}-\frac{2}{\frac{1}{x}+\frac{1}{y}}$$

3. (*a*) Solve:

$$2x-y+z=1$$
$$3x+y+z=2$$
$$x+y+z=0$$

(*b*) Simplify the following fraction by rationalizing its denominator;

$$\frac{1}{1+\sqrt{2}+\sqrt{3}}$$

4. (*a*) Multiply $x^{\frac{3}{4}}y^{\frac{3}{4}}+2+x^{\frac{3}{4}}y^{\frac{3}{4}}$ by $x^{\frac{1}{4}}y^{\frac{1}{4}}-1+x^{\frac{1}{4}}y^{\frac{1}{4}}$

(*b*) Find the square root of:

$$x^{\frac{8}{5}}-2a^{\frac{3}{5}}x^{\frac{11}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+a^{\frac{8}{5}}$$

5. A garrison of 1000 men having provisions for 60 days was reinforced after 10 days, and from that time the provisions lasted only 20 days. Find the number in the reinforcement.

6. If the numerator of a certain fraction be increased by one and its denominator diminished by one, its value will be one. If the numerator be increased by the denominator and the denominator be diminished by the numerator, its value will be four. Find the fraction.

Mathematics A2 (Quadratics and Beyond)

1. (*a*) Find to two decimal places the values of x which satisfy the following equation:

$$x - \frac{3-2x}{x} + 2 = 0$$

(*b*) Solve for x:

$$\frac{1}{x+a+b} = \frac{1}{x} + \frac{1}{a} + \frac{1}{b}$$

2. (*a*) Find the values of x which satisfy the following equation:

$$\sqrt{3x+10} - \sqrt{x-1} = \sqrt{2x-1}$$

(*b*) For what relation in r, s, t, will the equation

$$xr^2 + 2sx + t = 0$$

have equal roots? Prove your answer to be correct.

3. Solve the following pair of simultaneous equations:

$$x^2 + y^2 - 25 = 0$$
$$7x + y \; - 25 = 0$$

Draw graph illustrating each equation and their solution.

4. (*a*) Expand by binomial theorem and express each term of expansion in simplest form:

$$\left(a^{-\frac{2}{3}} - \frac{a^{\frac{1}{2}}}{2}\right)^5$$

(*b*) The sum of the first and fourth terms of an arithmetical progression is 19 and the third and sixth terms is 31. What is the first term?

5. A broker sells certain railroad shares for \$3240. A few days later, the price having fallen \$9 per share, he buys for the same sum five more shares than he had sold. Find the price and the number of shares transferred on each day.

Solution of the Foregoing Examination

1. **(a)** $(4x + 14)\ (5x + 3) - 24x - 64 = (4x + 3)\ (5x + 3)$

$$20x^2 + 82x + 42 - 24x - 64 = 20x^2 + 27x + 9$$
$$82x - 51x = 9 + 64 - 42$$
$$31x = 31$$
$$x = 1.$$ **Answer.**

(b) $(x^2-6)^2\ (x-a-b)-(x-a)^2\ (x-a-b)=2\ (a-b)$ $(x-a)\ (x-b)$

$x^3-3x^2b+3xb^2-x^2a+2xab-ab^2-b^3-x^3-3x^2a-3xa^2+a^3+bx^2-2xab+a^2b=2x^2a-2xa^2+2a^2b-2x^2b-2xb^2-2ab^2$

From which; $x=\dfrac{(a^2+b^2)\ (b-a)}{5b^2-a^2}$ Answer.

2. (a) $xy^3\ (15+xy)\ (5-9xy)$. Answer.
$xy\ (x+y+1)\ (x-y)$. Answer.
$4\ (x-3)\ (x^2-x+1)$. Answer.

(b) $\dfrac{xy}{y+x}+\dfrac{xy}{x+y}-\dfrac{2xy}{y+x}=0$. Answer.

3. (a)

$$\begin{array}{rl} 3x+y+z=2 & \\ x+y+z=0 & \\ \hline 2x=2 & \\ x=1. & \text{Answer.} \\ y=0. & \text{Answer.} \end{array} \qquad \begin{array}{rl} 2x-y+z=1 & \\ 3x+y+z=2 & \\ \hline 5x+2z=3 & \\ 5+2z=3 & \\ z=-1. & \text{Answer.} \end{array}$$

Subtracting;

(b) Multiply both numerator and denominator by $1-\sqrt{2}-\sqrt{3}$ giving; $\dfrac{1-\sqrt{2}-\sqrt{3}}{-4-2\sqrt{6}}$ Multiply both numerator and denominator of the last fraction by $-4+2\sqrt{6}$ giving;

$$\frac{(1-\sqrt{2}-\sqrt{3})\ (-4+2\ \sqrt{6})}{-8}=\frac{2-\sqrt{2}\sqrt{3}+\sqrt{2}}{4}$$

4. (a) $xy^{-1}-x^{\frac{3}{4}}y^{\frac{3}{4}}+x^{\frac{1}{2}}y^{-\frac{1}{2}}+2x^{\frac{1}{4}}y^{-\frac{1}{4}}+2x^{-\frac{1}{4}}y^{\frac{1}{4}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}-x^{-\frac{3}{4}}y^{\frac{3}{4}}+x^{-1}y-2.$

$xy^{-1}+2x^{\frac{1}{4}}y^{-\frac{1}{4}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}-x^{\frac{3}{4}}y^{-\frac{3}{4}}-2-x^{-\frac{3}{4}}y^{\frac{3}{4}}+x^{\frac{1}{2}}y^{-\frac{1}{2}}+2x^{-\frac{1}{4}}y^{\frac{1}{4}}+x^{-1}y$ Answer.

(b) $x^{\frac{4}{5}}-a^{-\frac{3}{5}}x^{\frac{7}{5}}+a^{\frac{4}{5}}$. Answer.

5. Let x denote portion of provisions one man receives.
Let y denote number of men in the reinforcement.

Then $60\ (1000x)=1$ or $x=\dfrac{1}{60000}$

and $10\ (1000x)+20x\ (1000+y)=1$

Substituting the value $x=\dfrac{1}{60000}$ in the last equation gives;

$y=1500$ men. Answer.

6. Let x denote the numerator and y the denominator.

Then $\dfrac{x+1}{9-1}=1$ and $\dfrac{x+y}{y-x}=4$

$x=3$ and $y=5$. Therefore the fraction is $\frac{3}{5}$. Answer.

Answers to Mathematics A2 (Quadratics and Beyond)

1. (*a*) The given equation may be reduced to;
$x^2 - 3 + 2x + 2x = 0$. From which $x = .65$ or -4.65. **Answer.**
Checks; $.42 + 2.66 - 3 = 0$ and $21.6 - 18.6 - 3 = 0$ (See page 30)

(*b*) The given equation may be reduced to;

$$a^2b + ab^2 + ax^2 + bx^2 + a^2x + b^2x + 2abx = 0$$

From which
$$a\,(x + b) + x\,(x + b) = 0$$
$$(a + x)\,(b + x) = 0$$
$$x = -a \text{ or } -b. \quad \textbf{Answer.}$$

The answer may be proved by substituting each of these values for x in the given equation.

2. Squaring both members and reducing gives;

$$3x + 10 - 2\sqrt{3x + 10}\,\sqrt{x - 1} + x - 1 = 2x - 1$$

$x + 5 = \sqrt{3x^2 + 7x - 10}$. Again squaring both members and reducing gives; $(2x + 7)\,(x - 5) = 0$.

$$x = -\tfrac{7}{2} \text{ or } 5. \quad \textbf{Answer.}$$

(*b*) Both roots are equal when $s^2 = rt$; then $t = \dfrac{s^2}{r}$

Proof: substitute $\dfrac{s^2}{r}$ for t in the given equation;

then; $rx^2 + 2sx = -\dfrac{s^2}{r}$ from which $x = -\dfrac{s}{r}$

3. $x = 4$ or 3 and $y = -3$ or 4.
$x^2 + y^2 = 25$ is the equation of a circle and $7x + y - 25 = 0$ is the equation of a straight line intersecting the circle in one point whose coördinates are $x = 3$ and $y = 4$, and in another point whose coördinates are $x = 4$ and $y = -3$.

4. (*a*) According to the binomial theorem. (See Higher Algebra by Fisher and Schwatt, Philadelphia, Pa., University of Pennsylvania.)

$$a^{-\frac{10}{3}} - \tfrac{5}{2}a^{\frac{2}{3}(4)}a^{\frac{1}{2}} + \tfrac{20}{8}a^{-\frac{2}{3}(3)}a - \tfrac{60}{48}a^{-\frac{2}{3}(2)}a^{\frac{3}{2}} + \tfrac{5}{16}a^{-\frac{2}{3}}a^{2} - \frac{a^{\frac{5}{2}}}{32}$$

Which reduces to;

$$a^{-\frac{10}{3}} - \tfrac{5}{2}a^{-\frac{13}{6}} + \tfrac{5}{2}a^{-\frac{5}{3}} - \tfrac{5}{4}a^{\frac{1}{6}} + \tfrac{5}{16}a^{\frac{4}{3}} - \frac{a^{\frac{5}{2}}}{32}. \quad \textbf{Answer.}$$

(*b*) Let x denote the first term, and y the common difference.

Then $\quad x + (x + 3y) = 19 \quad$ or $\quad 2x + 3y = 19$
and $\quad (x + 2y) + (x + 5y) = 31 \quad$ or $\quad \underline{2x + 7y = 31}$
From which; $x = 5$. **Answer.** Subtracting; $\quad -4y = -12$
$$y = 3$$

5. Let x denote price in dollars per share at first sale.
Let y denote number of shares.

Then
$$\left.\begin{aligned} xy &= 3240 \\ (x-9)(y+5) &= 3240 \end{aligned}\right\} = \left\{\begin{aligned} &xy = 3240 \\ &xy - 9y + 5x = 3825 \end{aligned}\right.$$

Substituting the value $y = \dfrac{3240}{x}$ in the last equation gives;

$$\frac{3240}{x} - 29160 + 5x^2 = 3285x$$

From which is obtained; $x^2 - 9x = 5832$. (See page 111)

$x = \$81.$ $y = 40$ **shares. Answer.**

If 5 more shares were purchased the second day the number must have been 45, and the price was $\dfrac{3240}{45} = \$72.$

Proof; $81 \times 40 = 3240 = 72 \times 45.$

WORCESTER POLYTECHNIC INSTITUTE

Entrance Examination

1. Algebra I. Wednesday, June 16, 1915. 2 to 3.35 P. M.

1. (*a*) Divide: $\frac{1}{3}a^5 + \frac{11}{45}a^4b - \frac{73}{30}a^3b^2 - \frac{31}{15}a^2b^3 + \frac{7}{10}ab^4 - 3b^5$ by $\frac{2}{3}a^2 - \frac{2}{5}ab - 4b^2$.

(*b*) Multiply; $x^{\frac{1}{2}} - 4^{\frac{1}{2}} + 6x^{-\frac{1}{2}}y - 4x^{-1}y^{\frac{2}{3}} + x^{-\frac{3}{2}}y^2$ by $x^{\frac{1}{2}}y^{-\frac{1}{2}} + 2 + x^{-\frac{1}{2}}y^{\frac{1}{2}}$.

2. Simplify: $$\frac{\dfrac{2a+b}{a+b} - 1}{1 - \dfrac{b}{a+b}} \quad \text{and} \quad \frac{1}{1 + \dfrac{c}{1 + c + \dfrac{2c^2}{1-c}}}$$

3. Solve the following equation for x:

$$\frac{2x+1}{2x-1} - \frac{8}{4x^2-1} = \frac{2x-1}{2x+1}$$

4. A merchant adds to his capital one-fourth of it each year. At the end of each year he deducts \$1200 for expenses. At the end of the third year he has, after the deduction of the last \$1200, one and a half times his original capital, minus \$950. What was his original capital?

5. Solve for x and y the simultaneous equations:

$$x - \frac{2y - x}{23 - x} = 20 + \frac{2x - 59}{2}$$

$$y - \frac{y - 3}{x - 18} = 30 - \frac{73 - 3y}{3}$$

Answers to the foregoing examination

1. $\frac{1}{2}a^3 + \frac{2}{3}a^2b - \frac{1}{4}ab^2 + \frac{3}{4}b^3$. **Answer.**

2. (a) $\dfrac{\dfrac{2a+b}{a+b} - 1}{1 - \dfrac{b}{a+b}} = \dfrac{\dfrac{2a+b}{a}}{a+b} = \dfrac{2a - a}{a} = \dfrac{a}{a} = 1.$ **Answer.**

(b) $\dfrac{1}{1 + \dfrac{c}{1 + c + \dfrac{2c^2}{1-c}}} = \dfrac{1}{1 + \dfrac{c}{\dfrac{1+c^2}{1-c}}} = \dfrac{1+c^2}{1-c}$ **Answer.**

3. $(2x + 1)^2 - 8 = (2x - 1)^2$

$$4x^2 + 4x - 7 = 4x^2 - 4x + 1$$

$$8x = 8$$

$x = 1.$ **Answer.**

4. See Example 29, page 86.

5. $2x(23 - x) - (2y - x)2 = 20(23 - x)2 + (2x - 59)\ (23 - x)$

Reduces to; $-17x - 4y = 437$

$3y\ (x - 18) - 3\ (y - 3) = 90\ (x - 18) - (73 - 3y)\ (x - 18)$

Reduces to; $-17x - 3y = -1629 + 1314 = -315$

$$\begin{aligned} 17x + 4y &= 437 \\ -17x - 3y &= -315 \\ \hline y &= 122, \end{aligned}$$

and $x = -3.$ **Answer.**

Let x denote his original capital.

Then $x + \frac{1}{4}x - \$1200$ will denote capital at the end of first year;

$(x + \frac{1}{4}x - 1200) + \dfrac{x + \frac{1}{4}x - 1200}{4} - \$1200$ denotes capital at end of second year; and $\frac{125}{64}x - 1875 - 1500 - 1200$ denotes capital at end of third year.

By condition stated in problem; $\frac{125}{64}x - 4575 = \frac{3}{2}x - 950.$

From which; $x = $ **\$8000; the original capital. Answer.**

WORCESTER POLYTECHNIC INSTITUTE

Entrance Examination

2. Algebra II. Wednesday, June 16, 1915; 3.45 to 5.15 P. M.

1. A man has a rectangular house which occupies 1200 sq. feet in the middle of a rectangular lot 8000 sq. feet in area. The lot extends 30 feet beyond the house at each end and 25 feet at each side. What are the dimensions of the lot?

2. Show that the equation $3x^2 + 5x + 1 - k^2 = 0$ has two real roots for any real value that may be given to k.

3. According to Boyle's Law, the volume of a gas is inversely proportional to the pressure on it. A tank contains 4 cu. ft. of air under a pressure of 60 lbs. per sq. in. How much space will be occupied by the air when expanded so as to be under atmospheric pressure, 15 lbs. per sq. in?

4. **(a)** Given that G is the Geometric Mean of two numbers a and b, that H is the Harmonic Mean of a and b, and that M is the Arithmetic Mean of a and b. Prove that G is also the Geometric Mean of H and M.

(b) The first term of an Arithmetic Progression is 1, the last term 7, and the sum of all the terms is 13. Find the Progression.

5. **(a)** Compute $(0.99)^5$ by expanding $(1 - .01)^5$ by the Binomial Theorem, and then simplifying.

(b) Find the fifth term of $\left(\sqrt{x} + \frac{1}{3\sqrt{x}}\right)^{10}$ Simplify it.

The foregoing examination is worked out as follows:

1. Let x denote the width, in feet, of the lot.

Then $\frac{8000}{x}$ will denote, in feet, the length of the lot.

By the given conditions; $(x - 50)\ (\frac{8000}{x} - 60) = 1200$

Which reduces to; $(x - 50)\ (8000 - 60x) = 1200x$

And performing the multiplication, to; $8000x - 60x^2 - 400000 + 3000x = 1200x$.

Giving; $x^2 - \frac{490}{3}x = -\frac{20000}{3}$

Completing the square by adding to both members of the equation the *square of ½ the coefficient of x*, gives;

$$x^2 = \tfrac{490}{3}x + (\tfrac{490}{6})^2 = -\tfrac{20000}{3} + (\tfrac{490}{6})^2$$

$$x - \tfrac{490}{6} = \pm\sqrt{(\tfrac{490}{6})^2 - \tfrac{20000}{3}}$$

$$x = \tfrac{490}{9} \pm \sqrt{\tfrac{240100}{36} - \tfrac{240000}{36}}$$

$$= \tfrac{490}{6} \pm \sqrt{\tfrac{100}{36}} = \tfrac{490}{6} \pm \tfrac{10}{6}$$

$$= \tfrac{480}{6} \text{ or } \tfrac{500}{6}$$

$$= 80 \text{ or } 83\tfrac{1}{3}$$

If the value of $x = 80$ **feet** as the width of the field is accepted, then the length will be $\frac{8000}{80}$ = **100 feet. Answer.**

If the width is taken as **83 $\frac{1}{3}$ feet**, the length will be **96 feet.**

Answer.

2. If the equation is solved for x the result is;

$$x = -\frac{5 \pm \sqrt{25 - 12 + 12k^2}}{6} = \frac{-5 \pm \sqrt{13 + 12k^2}}{6}$$

If the quantity expressed by the radical is positive the roots are *real;* if the value of the radical is negative the roots are unreal or *imaginary.* If k has any value whatever assigned to it the value of the radical is always positive and the equation will have two real roots.

3. According to Boyle's law $pv = k$.

Let v_1 denote the volume under 15 pounds pressure.

Then according to Boyle's law;

$$\frac{v_1}{4} = \frac{60}{15} \text{ from which } v_1 = \frac{60 \times 4}{15}$$ = **16 cubic feet. Answer.**

4. (a) Given the Geometric Progression $a + G + b$ (1)
Given the Harmonic Progression $a + H + b$ (2)
Given the Arithmetic Progression $a + A + b$ (3)

To show that $G^2 = A\,H$

From (3) $A - a = b - A$

From (2) $\frac{a}{b} = \frac{a - H}{H + b}$ and $a\,(H - b) = b\,(a - H)$

from which $H = \frac{2ab}{a + b}$

$$G^2 = A\,H \begin{cases} G^2 = ab \\ 2A = a + b \\ A = \dfrac{2ab}{a+b} \end{cases}$$

(b) See page 132. $1, \frac{11}{3}, \frac{19}{3}, \frac{27}{3}, \frac{35}{3}$. **Answer.**

5. **(a)** $(1-.01)^5=(1)^5-5\,(1)^4+10\,(1)^3\,(.01)^2-10\,(1)^2$ $(.01)^3+5\,(1)\,(.01)^4-(.01)^5$

$$=1-.05+.001-.00001+.00000005+.0000000001$$
$$=1.00100005-.\ .0500100001$$
$$=0.9509899499.\ \textbf{Answer.}$$

(b) The general expression for the r *th* term of $(a+b)^n=$

$$\frac{n\,(n-1)\,(n-2)\ldots(n-r+2)}{1\times2\times3\ldots(r-1)}\,a^{(n-r+1)}\,b^{(r-1)}$$

In the given case: $n=10$

$r=5$

$a=\sqrt{x}$

$b=\dfrac{1}{3\sqrt{x}}$

Substituting these values in the general expression gives;

$$\frac{10\,(9)\,(8)\,(7)}{1\times2\times3\times4}\,(\sqrt{x})^6\left(\frac{1}{3\sqrt{x}}\right)^4=\frac{210x^3}{81x^2}=\tfrac{70}{27}x.\ \ \textbf{Answer.}$$

YALE ENTRANCE EXAMINATION, JUNE, 1912

Algebra A. [Time allowed, one hour.]

[Omit one question in Group II and one question in Group III.]

Group I.

1. Resolve into prime factors;
   - (a) $6x^2-7x-20$
   - (b) $(x^2-5x)^2-2\,(x^2-5x)-24$
   - (c) a^4+4a^2+16

2. Simplify $\left(5-\dfrac{a^2-19x^2}{a^2-4x^2}\right)\div\left(3-\dfrac{a-5x}{a-2x}\right)$

3. Solve $\dfrac{2\,(x-7)}{x^2+3x-28}+\dfrac{2-x}{4-x}-\dfrac{x+3}{x+7}=0$

Group II.

4. Simplify $\dfrac{\sqrt{2}+2\sqrt{3}}{\sqrt{2}-\sqrt{12}}$ and compute the value of the fraction to two decimal places.

5. Solve the simultaneous equations $\begin{cases} x^{-\frac{1}{2}}+2y^{-\frac{1}{2}}=\frac{7}{6} \\ 2x^{-\frac{1}{2}}-y^{-\frac{1}{2}}=\frac{2}{3} \end{cases}$

Group III.

6. Two numbers are in the ratio of $c:d$. If a should be added to the first and subtracted from the second, the results will be in the ratio of $3:2$. Find the numbers.

7. A dealer has two kinds of coffee, worth 30 and 40 cents per pound respectively. How many pounds of each must be taken to make a mixture of 70 pounds, worth 36 cents per pound?

8. A, B, and C can do a piece of work in 30 hours. A can do half as much again as B, and B two-thirds as much again as C. How long would each require to do the work alone?

Answers to the foregoing examination:

1. (a) $(3x+4)\,(2x-5)$.

(b) $[\,(x^2-5x)-6]\,[\,(x^2-5x)+4] = (x-6)\,(x+1) \times (x-4)\,(x-1)$.

(c) a^4+4a^2+16. **Answer.**

2. $$\frac{5a^2-20x^2-a^2+19x^2}{a^2-4x^2} \div \frac{3a-6x-a+5x}{a-2x} = \frac{4a^2-x^2}{a^2-4x^2} \times \frac{a-2x}{2a-x}$$

$$=\frac{(2a-x)\,(2a+x)\,(a-2x)}{(a-2x)\,(a+2x)\,(2a-x)} = \frac{2a+x}{a+2x}.$$ **Answer**

3. $$\frac{(2x-14)+(x+7)\,(2-x)-(4-x)\,(x+3)}{-x^2-3x+28}$$

$$=\frac{2x-14-x^2-5x+14+x^2-x-12}{-x^2-3x+28}$$

$= 8x-16=0$; from which $x=2$. **Answer.**

4. $$\frac{\sqrt{2}+2\sqrt{3}}{\sqrt{2}-\sqrt{3}\sqrt{4}} = \frac{\sqrt{2}-2\sqrt{3}}{\sqrt{2}-\sqrt{3}\sqrt{2}\sqrt{2}} = \frac{1+\sqrt{2}\sqrt{3}}{1-\sqrt{2}\sqrt{3}}$$

$$=\frac{1+1.414\times1.732}{1-1.414\times1.732} = \frac{-3.449}{1.449} = 2.38.$$ **Answer.**

5. Multiply the first equation by 2, then subtract second from first.

$$\begin{array}{l} 2x^{-\frac{1}{2}}+4y^{-\frac{1}{2}}=\frac{14}{6}=\frac{7}{3} \\ 2x^{-\frac{1}{2}}-\ y^{-\frac{1}{2}}= \qquad \frac{2}{3} \\ \hline 5y^{-\frac{1}{2}}=\frac{5}{3} \end{array}$$

$$\frac{1}{\sqrt{y}}=\frac{5}{15}=\frac{1}{3}$$

$\sqrt{y}=3 \therefore y=9$. **Answer.**

$x=4$. **Answer.**

6. Let x denote one number and y denote the other; then;

$$\frac{x}{y}=\frac{c}{d}\text{ and } x=y\frac{c}{d}$$

and $\dfrac{x+a}{y-a}=\frac{3}{2}$ from which $2x+2a=3y-3a$, and $y=-\dfrac{5ad}{2c-3d}$

$x=\dfrac{c}{d}\times-\dfrac{5ad}{2c-3d}=-\dfrac{5ac}{2c-3d}$ **Answer.**

7. Let x denote the number of pounds at 30 cents.
Let y denote the number of pounds at 40 cents.
Then $30x+40y=70\times 36=2520$
and $x+y=70$. From which $x=28$ lbs. and $y=42$ lbs.
Proof: $42\times 40+28\times 30=2520$ cents.

8. Let x denote number of hours required by A to do the work alone.
Let y denote number of hours required by B to do the work alone.
Let z denote number of hours required by C to do the work alone.
Then; $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{30}$; also $x=\frac{2}{3}y$ and $y=\frac{3}{5}z$. $y=$ **93 hrs.** $x=$ **62 hrs.** $z=$ **155 hrs. Answer.**

Algebra B. [Time allowed, one hour.]

(Omit one question in Group I and one in Group II. Credit will be given for *five* questions only.)

Group I.

1. Solve $\dfrac{x+a}{x+b}+\dfrac{x+b}{x+a}=\dfrac{5}{2}$

2. Solve the simultaneous equations $\begin{cases} x^2y^2+28xy-480=0 \\ 2x+y=11 \end{cases}$

Arrange the roots in corresponding pairs.

3. Solve; $3x^{-\frac{3}{2}}+20x^{-\frac{3}{4}}=32.$

Group II.

4. In going 7500 yards a front wheel of a wagon makes 1000 more revolutions than a rear one. If the wheels were each a yard greater in circumference, a front wheel would make 625 more revolutions than a rear one. What is the circumference of each?

5. Two trolley cars of equal speed leave A and B, which are 20 miles apart, at different times. Just as the cars pass each other an

accident reduces the power and their speed is decreased 10 miles per hour. One car makes the journey from A to B in 56 minutes and the other from B to A in 72 minutes. What is their common speed?

Group III.

6. Write in the simplest form the last three terms of the expansion of $(4a^{\frac{3}{2}} - a^{\frac{1}{2}}x^{\frac{1}{3}})^8$

7. **(a)** Derive the formula for the sum of an arithmetical progression.

(b) Find the sum to infinity of the series $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\ldots$ Also find the sum of the positive terms.

Solution of the foregoing examination;

1. $2(x+a)(x+a) + 2(x+b)(x+b) = 5(x+b)(x+a)$

$$2(x^2+2ax+a^2) + 2(x^2+2bx+b^2) = 5(x^2+ax+bx+ab)$$

$$x^2 + ax - 2a^2 + bx + 5ab - 2b^2 = 0$$

$$x^2 + (a+b)x = 2a^2 - 5ab + 2b^2$$

$$x^2 + (a+b)x + \left(\frac{a+b}{2}\right)^2 = \frac{a^2+2ab+b^2}{4} + 2a^2 - 5ab + 2b^2$$

$$= \frac{a^2+2ab+b^2+8a^2-20ab+8b^2}{4}$$

$$= \frac{9a^2-18ab+9b^2}{4}$$

$$x = -\frac{a+b}{2} \pm \frac{\sqrt{9a^2-18ab+9b^2}}{2} = \frac{-a-b \pm 3a-3b}{2}$$

$= \dfrac{2a-4b}{2} = \mathbf{a-2b}$. **Answer.**

or $\dfrac{-4a+2b}{2} = \mathbf{2a+b}$. **Answer.**

2. The first equation may be expressed; $(xy-12)(xy+40) = 0$, from which $x = \frac{12}{y}$ or $-\frac{40}{y}$. By proper substitution in the second equation; $2x + \frac{12}{x} = 11$; from which $x = \frac{3}{2}$ or 4. Therefore $y =$ 8 or 3 and $x = \frac{3}{2}$ or 4. **Answer.**

3. The equation may be written as follows; (consult page 36)

$$\frac{3}{x^{\frac{3}{2}}} + \frac{20}{x^{\frac{3}{4}}} = 32 \text{ which is the same as } \frac{3}{\sqrt{x^3}} + \frac{20}{\sqrt[4]{x^3}} = 32.$$

The last equation may be reduced to;

$\sqrt{x^3} - \frac{5}{8}\sqrt[4]{x^3} - \frac{3}{32} = 0$, which is equal to $\sqrt{x^3} - \frac{5}{8}\sqrt[4]{x^3} + \frac{25}{256} = \frac{49}{256}$

The roots of which are; $\sqrt[4]{x^3} = \frac{5}{16} \pm \frac{7}{16} = \frac{3}{4}$ or $= -\frac{1}{8}$

From which $x^3 = (-\frac{1}{8})^4$ or $(\frac{3}{4})^4$. From which $x = \frac{1}{16}$ or $\frac{9}{16}$

4. Let x denote circumference of front wheels, in yards.
Let y denote circumference of rear wheels, in yards.

Then; $\dfrac{7500}{x} - \dfrac{7500}{y} = 1000$

$$\frac{7500}{x+1} - \frac{7500}{y+1} = 625$$

$x = \dfrac{7500y}{7500 + 1000y}$ from first equation above.

By substituting the last value of x in the second equation gives;

$$7y^2 - 32y - 15 = 0$$
$$(7y + 3)(y - 5) = 0$$

$y = -\frac{3}{7}$ or **5 yards**; circumference of rear wheels.
$x = $ **3 yards**; circumference of front wheels. **Answer.**

(Consult Example 63, page 116)

5. Let x denote common speed, and let y denote the distance from A that the accident happened. Then $20 - x$ will denote the distance from B that the accident happened.

Then; $\dfrac{y}{x} + \dfrac{20-y}{x-10} = \dfrac{14}{15}$ and $\dfrac{20-y}{x} + \dfrac{y}{x-10} = \dfrac{6}{5}$

From which $x =$ **25 miles per hour. Answer.**

6. The last three terms will be the 7th, 8th, and 9th. See page 164. solution of question **5** (***b***).

$n = 8$; $a = 4a^{\frac{3}{2}}$; $b = -a^{\frac{1}{2}}x^{\frac{1}{3}}$

7th $= 28\,(16)a^6x^2 = 448a^6x^2 =$ **Answer.**

8th $= -32a^5x^{\frac{7}{3}}$. **Answer.**

9th $= a^4x^{\frac{8}{3}}$. **Answer.**

7. (***a***) See page 135.

For finding the sum of decreasing series.

Rule. *Multiply the first term by the ratio, and divide the product by the ratio less 1.* The resulting quotient is the sum of an infinite decreasing series.

(*b*) In the given case the ratio is $1 \div -\frac{1}{2} = -2$.

Applying the above rule: $\dfrac{1 \times (-2)}{-3} = +\frac{2}{3}$. **Answer.**

Applying the same rule to the positive terms gives the ratio $= 1 \div \frac{1}{4} = 4$, and sum $= \dfrac{4 \times 1}{3} = \frac{4}{3}$. **Answer.**

The sum of the *negative* terms may also be found by the same rule. The ratio is 4, and sum $= \dfrac{-\frac{1}{2} \times 4}{3} = -\frac{2}{3}$

The sum of the *negative* terms may also be found by the same $\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$ which confirms the first answer given under (*b*).

INDEX

All references are to pages

Review from **"The Wireless World,"** June, 1916, London, Eng.

Directions for Designing, Making, and Operating High-Pressure Transformers, by Professor F. E. Austin, Hanover, N. H. 3s. net.

This is an interesting and clearly written little book, particularly valuable to the serious student of wireless and to the operator who is anxious to understand thoroughly the principles and construction of the component parts of his installation.

The author introduces the subject by referring to the commercial demand and necessity for electric power at high pressure, and the reasons why alternating current is the most useful for this purpose. A simple but very practical explanation of the construction of the transformer then follows, after which we find an explanation of symbols and annotation, the various losses in a transformer, power factor, and other matters. The author next treats of the design of a 20,000 volt transformer, entering very carefully into practical details of calculation. Following this, we have a chapter entitled "Directions and Data for Constructing a 3-KW. 20,000 volt Transformer," the approximate cost of materials not being overlooked. A further chapter deals with data applying to a 4,000-volt transformer.

We do not remember having previously seen any small book dealing so thoroughly and practically with the construction of high pressure transformers, nor one in which the diagrams and photographic illustrations were so happily chosen. The impression we have gained after reading the book is that the author knows exactly what he is talking about and how to express himself.

From **"Journal United States Artillery."**

Directions for Designing, Making, and Operating High-Pressure Transformers. By Professor F. E. Austin, Box 441, Hanover, N. H. 5″ x 7¾″. 46 pp. 21 il. Tab. Cloth. Price, 65 cents.

This seems to be a very practical little volume. For one who already has a knowledge of the fundamental principles of transformers and who has any reason to construct experimental apparatus of this character, the book will give all the practical and theoretical information desired and that, too, in a very small compass and in a very readable and clear form.

The illustrations are numerous and make the explanation very clear.

All the details of the mechanical process of winding, making up the core, etc., are simply explained and clearly illustrated.

All the necessary theoretical calculations are made evident by type examples worked out. All theoretical principles are simply and clearly explained. Specifications, drawings, and bills of material with approximate costs are given to guide the investigator.

The book should be a great help to an instructor, to a student working in an alternating current laboratory, or to an investigator making up transformers for experimental purposes.

Remit amount to:

PROF. F. E. AUSTIN, Box 441, Hanover, N. H.

Distributors for England, E. & F. N. SPON, Ltd., 57 Haymarket, London.

From "**Journal United States Artillery.**"

Examples in Alternating Currents. Vol. I. (For Students and Engineers.) By F. E. Austin, Box 441, Hanover, N. H. 5" x 7½". 223 pp. 4 plates, 70 figures. Price: $2.40.

The purpose of the book as stated by the author is to help students of electrical engineering gain a better knowledge of fundamental principles by solution of well selected problems involving those principles.

The problems are numerous, well chosen, and well arranged under proper headings. Each principle is illustrated by an example worked out in detail. Following each illustrative example are sufficient problems of the same general character to give the student adequate practice in testing his knowledge of the particular principles involved and the method of solution.

The first part of the book explains briefly, and illustrates the application of, trigonometry and differential and integral calculus to the solution of problems in alternating currents.

The problems which follow involve the use of trigonometry, differential calculus, integral calculus, the vector diagram, and algebra. For students who desire to limit their study to problems most frequently met with in practice and who do not desire to use the calculus, those in the latter part of the book furnish examples illustrating every principle involved in practical work, and the solutions are by vector diagrams or are algebraic.

The tables in the back of the book contain values of the variable quantities entering the alternating current formulas that will save much work in solving problems.

The book is of value to a student of alternating-currents who desires to ground himself in the fundamentals of the subject. An instructor will find it of great value in furnishing a well selected and classified list of problems suitable for tests or illustrative use.

Review by "**Mining and Scientific Press.**"

This work has been carefully prepared, and is published in pocket size. It contains examples dealing with wave-length, frequency, sine and nonsine alternating-currents and pressures, generators, power, power-factor, capacity, inductance, and resonance, showing the process of solution, step by step, together with the fundamental equations applying, and the necessary data properly arranged. The diagrams are very clear, greatly assisting in an understanding of the fundamental principles. Electrical engineers will find the trigonometric formulæ of value, and tables of variable quantities for frequencies from 1 to 150,000 cycles.

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